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Zonal Harmonic Perturbations of an Accurate Reference Orbit of an Artificial Satellite*

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Abstract

The theory developed in an earlier paper, for an accurate reference orbit of an artificial satellite, is first slightly modified, so as to prepare the way for a treatment of zonal harmonic perturbations. Delaunay variables are next introduced, by means of certain linear combinations of the action variables, along with their canonical conjugates. Application of the von Zeipel method then permits the calculation of the most important zonal harmonic perturbations. These arise from the third, with coefficient J_3, and the residual fourth, with coefficient $J_4 + J_2^2$. The accuracy of the secular and short-periodic effects is through terms of order J_2^2 and that of the long-periodic effects is through terms of order J_2. Since the reference orbit itself, with exact secular terms, takes care of all but 0.5 percent of the deviation of the earth's gravitational field from spherical symmetry, the over-all secular accuracy ~~is much greater than~~ surpasses that of other second order theories. The results are compared with those of Kozai.

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1. The Reference Orbit

The author (Vinti 1959a,b) has introduced a potential

$$V' = -\mu\rho(\rho^2 + c^2\eta^2)^{-1} \tag{1.00}$$

that can represent accurately the gravitational field of an oblate planet. Here μ is the product of the gravitational constant G and the planet's mass, c is an adjustable length, and ρ and η are oblate spheroidal coordinates, defined by the equations

$$X + iY = r\cos\theta \exp i\phi = [(\rho^2 + c^2)(1 - \eta^2)]^{\frac{1}{2}} \exp i\phi \tag{1.01}$$

$$Z = r\sin\theta = \rho\eta \tag{1.02}$$

If an artificial satellite is at the field point, r, θ, and ϕ are respectively its planetocentric distance, declination, and right ascension, and X, Y, and Z are its rectangular coordinates, OZ being along the planet's axis and OX pointing toward its vernal equinox.

If r_e is the equatorial radius, the true potential is

$$V = -\mu r^{-1}[1 - \sum_{n=2}^{\infty} (r_e/r)^n J_n P_n(\sin\theta)] + \text{tesseral harmonics}, \tag{1.03}$$

where only the products $r_e^n J_n$ need to be known. That is, differences in the definition of r_e, when non-circularity of the equator is taken into account, can be reconciled by small adjustments of the J's. Then if

$$c = r_e J_2^{1/2}, \tag{1.04}$$

V' represents exactly the zeroth harmonic $-\mu/r$ and the second harmonic and also gives higher even harmonics, characterized by

$$J_{2m} = (-1)^{m+1} J_2^m \tag{1.05}$$

In particular it gives $J_4 + J_2^2 = 0$, as compared with observed values for the earth ranging from $-(0.9)10^{-6}$ to $(0.4)10^{-6}$ (Kaula 1962, King-Hele, Cook, and Rees 1963). Consequently it accounts for about 99.5% of the deviation of V from the value $-\mu/r$ corresponding to spherical symmetry. It thus accounts almost completely for the flattening of the earth, leading to a geoid that never departs by more than about 30 meters from the true sea-level surface.

For the drag-free motion of an artificial satellite the potential (1.00) leads to a separable problem, which has been worked out analytically (Vinti 1961 a,b, 1962). This solution, holding for all angles of inclination and containing no critical inclination or long-periodic terms, gives secular terms exactly by means of rapidly converging infinite series and short-periodic terms correctly through order J_2^2. We call this orbit corresponding to (1.00) the reference orbit. For such a reference orbit error can never accumulate, because of the exactness of the secular terms, and the periodic terms can be in error only by amounts of the order J_2^3, i.e., by about 1 part in 10^9, since $J_2 = (1.08)10^{-3}$ for the earth.

2. Zonal Harmonic Perturbations

For a satellite of the earth, if its orbit is high enough so that drag is small and low enough so that the moon's effect is small, the above reference orbit ought to hold rather well for a good many revolutions. (I purposely choose vague words here, since numerical comparisons are still incomplete.) Eventually, however, the actual orbit will deviate more and more from such a reference orbit, because of the neglected forces. These include forces arising from drag, meteoritic impact, radiation, electromagnetic fields, the sun and the moon, and the neglected part of the earth's gravitational potential, corresponding to (1.03) minus (1.00). Since the expansion of (1.00) in zonal harmonics is

$$V' = -\mu r^{-1} \sum_{m=1}^{\infty} (r_e/r)^{2m} (-J_2)^m P_{2m}(\sin\theta), \qquad (2.00)$$

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this difference is

$$V - V' = \mu r^{-1}[(\frac{r_e}{r})^3 J_3 P_3(\sin\theta) + (\frac{r_e}{r})^4 (J_4 + J_2^2) P_4(\sin\theta) + (\frac{r_e}{r})^5 J_5 P_5(\sin\theta)$$

$$+ (\frac{r_e}{r})^6 (J_6 - J_2^3) P_6(\sin\theta) + \ldots] + \text{tesseral harmonics} \qquad (2.01)$$

Of these forces the most important, for any satellite with a large ratio of mass to area, are the forces corresponding to J_3 and $J_4 + J_2^2$ in (2.01) and drag, which as determined empirically may include effects of meteoritic impact. For a double satellite (Langer and Vinti 1963) only (2.01) and the lunar-solar perturbation remain.

The purpose of the present paper is to devise a method for correcting for the effects of any of the zonal harmonics in (2.01). The first example considered is the residual fourth harmonic, with coefficient $J_4 + J_2^2$. This harmonic leads not only to short-periodic effects and secular effects, but also to long-periodic effects depending on a resonance denominator $1-5 \cos^2 I$, giving rise to a critical inclination $I = 63.4°$. The second example considered is the third harmonic, with coefficient J_3. This gives rise only to short-periodic effects and to long-periodic effects without singularities, so that it is qualitatively less interesting. Because of its greater magnitude, however, J_3 being about $(-2.4)10^{-6}$ and $|J_4 + J_2^2|$ being probably somewhat less than $(0.5)10^{-6}$ (Kaula 1962, King-Hele, Cook, and Rees 1963), it leads to somewhat larger periodic effects.

3. The Dynamical Problem

Our problem is thus to find the motion of a satellite, taken to be of unit mass, when the Hamiltonian is

$$F = - T + \mu\rho(\rho^2 + c^2\,\eta^2)^{-1} + F_1 \,, \tag{3.01}$$

where T is its kinetic energy and where, for the residual fourth harmonic

$$F_1 = -\mu\, r_e^4\, r^{-5}\, \sigma_4 P_4(\sin\theta) \tag{3.02}$$

$$\sigma_4 \equiv J_4 + J_2^2 \tag{3.03}$$

(We have here reversed the sign of the Hamiltonian, to agree with the usual practice with Delaunay variables.) In carrying out this solution we shall use the results and notation of the solution (Vinti 1961 a,b, 1962) for the reference orbit, for which $F' = 0$. If in (3.02) we then put

$$r = a(1 - e\cos E) = a(1 - e^2)(1 + e\cos v)^{-1} \tag{3.04}$$

$$\sin\theta = \sin I \sin(v + \beta_2) \,, \tag{3.05}$$

the expression (3.02) for F_1 will be correct through order J_2^2. To this order of accuracy ~~the orbital elements in (3.04) and (3.05), viz., a, e, I, and β_2 may correspond either to the reference orbit or to an elliptic orbit and~~ the anomalies E and v may be given by the quasi-elliptic expressions

$$E \approx M_s + E_0 \qquad (3.06) \qquad\qquad v \approx M_s + v_0 \qquad (3.07)$$

Note that (3.04) corresponds to $r \approx p$ and (3.05) to other approximations of zeroth order in J_2, viz.,

$$\psi \approx \psi_s + \psi_0 \approx M_s + \beta_2 + v_0 \approx v + \beta_2 \qquad (3.08) \qquad \eta \approx \sin\theta \qquad (3.09)$$

Such an order of accuracy will result in errors of order J_2^3 for those secular and short-periodic effects which are produced by the perturbing potential (3.02) and of order J_2^2 for the corresponding long-periodic effects. This perturbation (3.02) represents about 0.1 % of the departure of the earth from sphericity. The solution for the other harmonics in(2.01) will have the same accuracy. However, since all of these higher harmonics represent only about 0.5% of the earth's departure from sphericity, their lower accuracy, as compared with that of the reference orbit which has already accounted for 99.5 % of this departure, should not result in serious cumulative errors.

In doing the perturbation theory, the first canonical variables that come to mind are the Jacobi "constants", viz., the α's and β's of the reference orbit. When the reference orbit is elliptic, however, their shortcomings are well known and they lead to the same troubles in the present problem, giving rise to Poisson terms, linear in the time, in the changes in α_1 and α_2.

The next set of canonical variables that one might try is the set generated from the α's and β's by the generating function

$$S' = -\alpha_1 t + \mu(-2\alpha_1)^{-\frac{1}{2}} \beta_1' + \alpha_2 \beta_2' + \alpha_3 \beta_3' \tag{3.10}$$

If we define n_0 by

$$\mu = n_0^2 a_0^3 \qquad a_0 \equiv -\tfrac{1}{2}\mu/\alpha_1 , \tag{3.11}$$

the resulting canonical variables are

$$\begin{array}{ll} L = (\mu a_0)^{\frac{1}{2}} & \ell = n_0(t + \beta_1) \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 , \end{array} \tag{3.12}$$

canonical with respect to the Hamiltonia

$$F = \tfrac{1}{2}\mu^2/L^2 + F_1 \tag{3.13}$$

When the reference orbit is elliptic, this set is the same as the fast Delaunay set (Garfinkel 1960).

One may then attempt to apply the von Zeipel method in the way successfully used by Brouwer (1959) and by Garfinkel (1959), first eliminating short-periodic terms and then proceeding to eliminate long-periodic terms. One finds, however, that the corresponding generating function S_1^*, which ought to be of the first order in the parameter $q_4 \equiv J_4 + J_2^2$, must then satisfy

$$\partial S_1^*/\partial \beta_1' = \text{zeroth order in } q_4 \qquad (3.14)$$

One may alternatively eliminate short-periodic and long-periodic terms simultaneously, but one then obtains a Poisson term of the form $v' \sin 2\beta_2'$ in $\alpha_2 - \alpha_2'$. Since v' has a secular part, such a result would appear absurd, since the "constant" α_2, which ought to have only a small periodic variation, would then increase indefinitely with time.

These difficulties are examples of the failure of the von Zeipel method whenever the following conditions <u>both</u> hold: (1) the perturbing potential has a long-periodic part of the first order in the perturbation parameter, and (2) the canonical variables are such that the unperturbed Hamiltonian depends only on L.

To obtain a successful set of variables, we may proceed as follows. Let q_i^o and p_i^o, $i = 1,2,3$, be the coordinates and momenta ρ, η, ϕ, p_ρ, p_η, p_ϕ corresponding to the unperturbed problem (the reference orbit), with Hamiltonian $F = F_0$. Also let j_i^o, w_i^o, $i = 1,2,3$, be the corresponding action and angle variables.

Then

$$p_i^o = \partial S(q_1^o,\ q_2^o,\ q_3^o,\ \overset{j_1^o}{\cancel{\alpha_1^o}},\ \overset{j_2^o}{\cancel{\alpha_2^o}},\ \overset{j_3^o}{\cancel{\alpha_3^o}})/\partial q_i^o \tag{3.15}$$

$$w_i^o = \partial S(q_1^o,\ q_2^o,\ q_3^o,\ \overset{j_1^o}{\cancel{\alpha_1^o}},\ \overset{j_2^o}{\cancel{\alpha_2^o}},\ \overset{j_3^o}{\cancel{\alpha_3^o}})/\partial j_i^o\ , \qquad (i=1,2,3) \tag{3.16}$$

where S is the Hamilton-Jacobi function of the unperturbed problem (Vinti 1959b), with the Jacobi α's replaced by the j_i^o. Here

$$j_i^o \equiv \oint p_i^o\, dq_i^o = j_i^o(\alpha_1,\ \alpha_2,\ \alpha_3) \tag{3.17}$$

Now let q_i, p_i, $i = 1,2,3$, be the coordinates and momenta corresponding to the perturbed problem, with Hamiltonian $F = F_0 + F_1$. Introduce new variables j_i, w_i, $i = 1,2,3$, by means of the canonical transformation

$$p_i = \partial S(q_1,\ q_2,\ q_3,\ \overset{j_1}{\cancel{p_1}},\ \overset{j_2}{\cancel{p_2}},\ \overset{j_3}{\cancel{p_3}})/\partial q_i \tag{3.18}$$

$$w_i = \partial S(q_1,\ q_2,\ q_3,\ j_1,\ j_2,\ j_3)/\partial j_i\ , \qquad (i=1,2,3) \tag{3.19}$$

where S is the same function of the q_i and j_i that the above Hamilton-Jacobi function is of the q_i^o and j_i^o. Then the w_i and j_i are canonical variables, satisfying the equations

$$dj_i/dt = \partial F/\partial w_i \neq 0 \tag{3.20}$$

$$dw_i/dt = -\partial F/\partial j_i \neq \text{ constant} \tag{3.21}$$

They are thus not action and angle variables, since the j_i are not constant and the w_i are not linear functions of t. Moreover

$$j_i \neq \oint p_i \, dq_i \, , \tag{3.22}$$

in contradistinction with (3.17).

It pays to go further, however, and introduce still another set of variables, a new Delaunay set L, G, H, ℓ, g, h, by the transformations

$$\begin{aligned} 2\pi L &= j_1 + j_2 + j_3 \operatorname{sgn}\alpha_3 \\ 2\pi G &= j_2 + j_3 \operatorname{sgn}\alpha_3 \\ 2\pi H &= j_3 \end{aligned} \tag{3.23}$$

$$\begin{aligned} \ell &= 2\pi w_1 \\ g &= 2\pi(w_2 - w_1) \\ h &= 2\pi(w_3 - w_2 \operatorname{sgn}\alpha_3) \, , \end{aligned} \tag{3.24}$$

where $\operatorname{sgn}\alpha_3 = \pm 1$ respectively for a direct orbit or a retrograde orbit. To verify that they are canonical, note that

$$Ld\ell + Gdg + Hdh = j_1\, dw_1 + j_2\, dw_2 + j_3\, dw_3 \qquad (3.25)$$

They were ~~introduced~~ used by Izsak (1962) in his application of the author's theory to the problem of the critical inclination. From (3.23) we now have

$$j_1 = 2\pi(L - G) \qquad j_2 = 2\pi(G - H\,\mathrm{sgn}\,\alpha_3) \qquad j_3 = 2\pi H \qquad (3.26)$$

4. The New Delaunay Set

The functional relations among any of the quantities α_i, j_i, w_i for the perturbed problem are the same as those connecting α_i^o, j_i^o, w_i^o for the unperturbed problem. We may therefore usually drop superscript zeros and depend on the context for the meanings of the quantities.

From the author's paper (Vinti 1959 b) we now find

$$j_1 = 2\int_{\rho_1}^{\rho_2} \frac{\partial S}{\partial \rho}\, d\rho = 2\pi[\mu(-2\alpha_1)^{-\frac{1}{2}} - \alpha_2] + O(J_2) \qquad (4.00)$$

$$j_2 = 4\int_{0}^{\eta_0} \frac{\partial S}{\partial \eta}\, d\eta = 2\pi(\alpha_2 - \alpha_3\,\mathrm{sgn}\,\alpha_3) + O(J_2) \qquad (4.01)$$

$$j_3 = 2\pi\,\alpha_3 \qquad (4.02)$$

Next put

$$\alpha_{rs} \equiv \partial\alpha_r/\partial j_s \quad (4.03) \qquad \nu_i \equiv \partial\alpha_i^o/\partial j_i^o \quad (4.04) \qquad j_{rs} \equiv \partial j_r/\partial\alpha_s \quad (4.05)$$

Then, since

$$\sum_{i=1}^{3} \frac{\partial S}{\partial j_i} \frac{\partial j_i}{\partial\alpha_k} = \frac{\partial S}{\partial\alpha_k} \quad , \qquad (k = 1,2,3) \qquad (4.06)$$

we find

$$\frac{\partial S}{\partial\alpha_1} = t + \beta_1 = j_{11}\, w_1 + j_{21}\, w_2 \qquad (4.07)$$

$$\frac{\partial S}{\partial\alpha_2} = \beta_2 = j_{12}\, w_1 + j_{22}\, w_2 \qquad (4.08)$$

$$\frac{\partial S}{\partial\alpha_3} = \beta_3 = j_{13}\, w_1 + j_{23}\, w_2 + 2\pi\, w_3 \qquad (4.09)$$

With the aid of (3.24) these equations become

$$2\pi(t + \beta_1) = j_{11}\ell + j_{21}(\ell + g) \qquad (4.10)$$

$$2\pi\, \beta_2 = j_{12}\ell + j_{22}(\ell + g) \qquad (4.11)$$

$$2\pi\, \beta_3 = j_{13}\ell + (j_{23} + 2\pi \operatorname{sgn} \alpha_3)(\ell + g) + 2\pi h \qquad (4.12)$$

The quantities j_{rs} occurring here are given explicitly as functions of the α's in Eqs. (7.16) through (7.21) of an earlier paper (Vinti 1961 a)

The constant orbital elements in the perturbed problem are then the constant parts a", e", η_0'' of a, e, η_0, along with the initial values ℓ_0'', g_0'', h_0'' of the secular parts of ℓ, g, h. The corresponding Hamiltonian F is given by

$$F = F_0(L,G,H) + F_1 \qquad (4.13) \qquad F_0 = -\alpha_1 , \qquad (4.14)$$

so that

$$\dot{L} = \partial F/\partial \ell \qquad\qquad \dot{\ell} = -\partial F/\partial L$$

$$\dot{G} = \partial F/\partial g \qquad (4.15) \qquad \dot{g} = -\partial F/\partial G \qquad (4.16)$$

$$\dot{H} = \partial F/\partial h = 0 \qquad\qquad \dot{h} = -\partial F/\partial H$$

With the use of (3.23), (3.26), and (4.04), we find for the <u>unperturbed problem</u> that

$$\dot{\ell} = -\frac{\partial F_0}{\partial L} = \frac{\partial \alpha_1}{\partial L} = \sum_{i=1}^{3} \frac{\partial \alpha_1}{\partial j_i}\frac{\partial j_i}{\partial L} = 2\pi\, \nu_1$$

$$\dot{g} = -\frac{\partial F_0}{\partial G} = \frac{\partial \alpha_1}{\partial G} = \sum_{i=1}^{3} \frac{\partial \alpha_1}{\alpha j_i}\frac{\partial j_i}{\partial G} = 2\pi(\nu_2 - \nu_1) \qquad (4.17)$$

$$\dot{h} = -\frac{\partial F_0}{\partial H} = \frac{\partial \alpha_1}{\partial H} = \sum_{i=1}^{3} \frac{\partial \alpha_1}{\partial j_i}\frac{\partial j_i}{\partial H} = 2\pi(\nu_3 - \nu_2 \operatorname{sgn} \alpha_3)$$

and thus for the unperturbed problem that

$$\begin{aligned} \ell &= \ell_0 + 2\pi\, \nu_1 t \\ g &= g_0 + 2\pi(\nu_2 - \nu_1)t \\ h &= h_0 + 2\pi(\nu_3 - \nu_2 \operatorname{sgn} \alpha_3)t \quad , \end{aligned} \tag{4.18}$$

where we have dropped the double primes from ℓ_0, g_0, h_0 .

Before finding the effects of the perturbing potential, it is desirable to change the algorithm for the unperturbed problem, given in an earlier paper (Vinti 1961 a, pp 197-200), so that the constant orbital elements become a,e,I, ℓ_0, g_0, and β_3 . (β_3 is better than h_0, as we shall see later.) To do so, insert (4.10), (4.11), and (4.18) into Eqs. (8.2) and (8.3) of that paper and carry out the same process that was carried out there. One finds

$$M_s = \ell_0 + 2\pi\, \nu_1 t \tag{4.19}$$

$$\psi_s = \ell_0 + g_0 + 2\pi\, \nu_2 t \tag{4.20}$$

With these new definitions of M_s and ψ_s the algorithm then becomes the same as in the earlier paper. Note, however, that the restriction on the angle of inclination I has been removed (Vinti 1962).

5. Variations in the Spheroidal Coordinates

It is convenient to derive here the variations in ρ, η, and ϕ that will arise from the variations produced by the perturbing potential in the Delaunay variables. From the later solution of the canonical equations (4.15) and (4.16) we shall find that the mth zonal harmonic produces variations in the Delaunay variables of the forms

$$\begin{aligned} \delta L &= \sigma_m L_m \\ \delta G &= \sigma_m G_m + \sigma_m J_2^{-1} \tilde{G}_m \\ \delta H &= 0 \end{aligned} \qquad (5.00)$$

$$\begin{aligned} \delta \ell &= \sigma_m \ell_m + \sigma_m J_2^{-1} \tilde{\ell}_m \\ \delta g &= \sigma_m g_m + \sigma_m J_2^{-1} \tilde{g}_m \\ \delta h &= \sigma_m h_m + \sigma_m J_2^{-1} \tilde{h}_m \end{aligned} \qquad (5.01)$$

Here $\sigma_3 = J_3$ and $\sigma_4 = J_4 + J_2^2$. The terms $\sigma_m L_m$ and $\sigma_m G_m$ are short-periodic of order J_2^2; the products of $\sigma_m J_2^{-1}$ with $\tilde{G}_m$, $\tilde{\ell}_m$, $\tilde{g}_m$, and $\tilde{h}_m$ are long-periodic of order J_2. For $m = 3$ the products of σ_3 with ℓ_3, g_3, and h_3 are short-periodic of order J_2^2. For $m = 4$

$$\begin{aligned} \ell_4 &= \ell_{41} + \ell_{42} \\ g_4 &= g_{41} + g_{42} \\ h_4 &= h_{41} + h_{42} , \end{aligned} \qquad (5.02)$$

where the products of σ_4 with ℓ_{41}, ~~ℓ_{42}~~ g_{41}, and ~~ℓ_{43}~~ h_{41} are short-periodic terms of order J_2^2 and where the products of σ_4 with ℓ_{42}, g_{42}, and

h_{42} are secular terms of order J_2^2 .

From δL and δG we can find the variations of the j's, then of the α's, and finally of the elements a,e, and $\eta_0 \equiv \sin I$. From these and from $\delta\ell$, δg, and δh we can then find δE, δv, $\delta\psi$, and $\delta\chi$ and finally the coordinate variations $\delta\rho$, $\delta\eta$, and $\delta\phi$. To obtain the δj's, use (3.26) and (5.00). The results are

$$\begin{aligned} \delta j_1 &= 2\pi(\sigma_m L_m - \sigma_m G_m - \sigma_m \tilde{G}_m/J_2) \\ \delta j_2 &= 2\pi(\sigma_m G_m + \sigma_m \tilde{G}_m/J_2) \qquad (5.03) \\ \delta j_3 &= 0 \end{aligned}$$

To find the $\delta\alpha$'s use

$$\delta\alpha_r = \sum_{i=1}^{3} \frac{\partial\alpha_r}{\partial j_i} \delta j_i \qquad (5.04)$$

Within the accuracy of the calculation, the coefficients $\partial\alpha_r/\partial j_i$ are needed only through order J_2^0 . Thus by (5.03), (5.04), and (4.04)

$$\delta\alpha_1 = 2\pi\nu_1\sigma_m L_m + 2\pi(\nu_2-\nu_1)\sigma_m G_m + 2\pi(\nu_2-\nu_1)\sigma_m \tilde{G}_m/J_2 \qquad (5.05)$$

Then, by (5.03), (5.04), and (4.03)

$$\delta\alpha_2 = 2\pi\alpha_{21}\sigma_m L_m + 2\pi(\alpha_{22}-\alpha_{21})\sigma_m G_m + 2\pi(\alpha_{22}-\alpha_{21})\sigma_m \tilde{G}_m/J_2 \qquad (5.06)$$

Also, by (4.02) and (5.03)

$$\delta\alpha_3 = 0 \qquad (5.07)$$

To find α_{21} and α_{22} note that

$$\sum_{i=1}^{3} \frac{\partial \alpha_2}{\partial j_i} \frac{\partial j_i}{\partial \alpha_2} = \frac{\partial \alpha_2}{\partial \alpha_2} = 1$$

$$\sum_{i=1}^{3} \frac{\partial \alpha_2}{\partial j_i} \frac{\partial j_i}{\partial \alpha_1} = \frac{\partial \alpha_2}{\partial \alpha_1} = 0 , \tag{5.08}$$

so that

$$\alpha_{21} j_{21} + \alpha_{22} j_{22} = 1$$

$$\alpha_{21} j_{11} + \alpha_{22} j_{12} = 0 , \tag{5.09}$$

with the solution

$$\alpha_{21} = - j_{21}/\Delta \qquad \alpha_{22} = j_{11}/\Delta , \tag{5.10}$$

where

$$\Delta \equiv j_{11} j_{22} - j_{12} j_{21} \tag{5.11}$$

From an earlier paper (Vinti 1961 a, p 189), it then follows that

$$\alpha_{21} = 0(J_2) \qquad \alpha_{22} = (2\pi)^{-1} + 0(J_2) \tag{5.12}$$

and also that

$$2\pi\nu_1 = n + 0(J_2) \tag{5.13}$$

$$\nu_2 - \nu_1 = 0(J_2) \ ,$$

where

$$n \equiv \mu^{\frac{1}{2}} \, a^{-3/2} \tag{5.14}$$

On carrying secular and short-periodic variations only through order J_2^2 and long-periodic perturbations only through order J_2, it then follows from (5.04) through (5.08) and (5.12) through (5.14) that

$$\delta\alpha_1 = n\sigma_m L_m \tag{5.15}$$

$$\delta\alpha_2 = \sigma_m G_m + \sigma_m \tilde{G}_m / J_2 \tag{5.16}$$

$$\delta\alpha_3 = 0 \tag{5.17}$$

In finding the variations of other quantities we may drop all terms in their defining equations of order J_2^k, where $k \geqq 1$. To show this, note that if $P = 0(J_2^0)$ and $\delta P = \sigma_m P_1 + \sigma_m J_2^{-1} \tilde{P}_1$ where P_1 and $\tilde{P}_1$ are both of order J_2^0, then

$$\delta(J_2^k P) = J_2^k(\sigma_m P_1 + \sigma_m J_2^{-1} \tilde{P}_1) \tag{5.18}$$

Thus if $k \geqq 1$, $J_2^k \sigma_m P_1$ is a secular plus short-periodic term of order $k + 2 \geqq 3$ and $J_2^k \sigma_m J_2^{-1} \widetilde{P}_1$ is a long-periodic term of order $k + 1 \geqq 2$. This proves the statement.

To obtain the variations of a,e, and η_0 we may thus use

$$a \approx - \frac{\mu}{2\alpha_1} \qquad 1 - e^2 \approx - \frac{2\alpha_1 \alpha_2^2}{\mu^2} \qquad 1 - \eta_0^2 \approx \frac{\alpha_3^2}{\alpha_2^2} \quad , \tag{5.19}$$

the neglected terms being of order J_2. There follow

$$\delta a = \frac{\mu}{2\alpha_1^2} \delta\alpha_1 = \frac{2a^2}{\mu} \delta\alpha_1$$

$$e\delta e = \frac{\alpha_2^2}{\mu^2} \delta\alpha_1 + \frac{2\alpha_1\alpha_2}{\mu^2} \delta\alpha_2 \tag{5.20}$$

$$\eta_0 \delta\eta_0 = \frac{\alpha_3^2}{\alpha_2^3} \delta\alpha_2$$

With use of (5.15), (5.16), (5.19), (5.20) and the relation

$\alpha_2^2 = \mu p + O(J_2)$, we then find

$$\delta a = \frac{2\sigma_m L_m}{an} \tag{5.21}$$

$$\delta e = \frac{pn}{\mu e} \sigma_m L_m - (ae)^{-1}\left(\frac{p}{\mu}\right)^{\frac{1}{2}}(\sigma_m G_m + \sigma_m J_2^{-1} \widetilde{G}_m) \tag{5.22}$$

$$\delta\eta_0 = \eta_0^{-1}(\mu p)^{-\frac{1}{2}}(1 - \eta_0^2)(\sigma_m G_m + \sigma_m J_2^{-1} \widetilde{G}_m) \tag{5.23}$$

The element $\underline{a}$ thus has only a short-periodic variation while the elements $\underline{e}$ and $\eta_0 \equiv \sin I$ have both short-periodic and long-periodic but no secular variations.

To find δE, δv, and $\delta\psi$ we first insert (4.10) and (4.11) into Eqs. (8.2) and (8.3) of the earlier paper (Vinti 1961 a), rejecting all terms of order J_2^k, where $k \geqq 1$. We find

$$E - e\sin E = \ell + O(J_2) \tag{5.24}$$

$$\psi = v + g + O(J_2) \tag{5.25}$$

Eq. (5.24) gives

$$\delta E = (1 - e\cos E)^{-1}(\delta\ell + \sin E\ \delta e) \tag{5.26}$$

To find δv, we use the anomaly connection

$$\tan\frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan\frac{E}{2}\ , \tag{5.27}$$

from which there follows

$$\delta v = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\left(1 - \frac{2e}{1+e}\sin^2\frac{v}{2}\right)\delta E + \frac{\sin v}{1-e^2}\,\delta e$$

$$= (1-e^2)^{\frac{1}{2}}(1-e\cos E)^{-1}\left[\delta E + (1-e^2)^{-1}\sin E\ \delta e\right] \tag{5.28}$$

Then

$$\delta\psi = \delta v + \delta g \tag{5.29}$$

The variations of the spheroidal coordinates

$$\rho = a(1-e\cos E) \tag{5.30}$$

$$\eta = \eta_0 \sin\psi \tag{5.31}$$

are then

$$\delta\rho = (1-e\cos E)\delta a - a\cos E\ \delta e + ae\sin E\ \delta E \tag{5.32}$$

$$\delta\eta = \sin\psi\ \delta\eta_0 + \eta_0\cos\psi\ \delta\psi \tag{5.33}$$

To find the variation $\delta\phi$ of the right ascension, we note that by Eq. (8.49) of the earlier paper (Vinti 1961 a),

$$\phi = \beta_3 + \chi\,\mathrm{sgn}\,\alpha_3 + O(J_2)\ , \tag{5.34}$$

where

$$\tan\chi = (1-\eta_0^2)^{\frac{1}{2}}\tan\psi \tag{5.35}$$

$$\cos\chi = (1-\eta_0^2\sin^2\psi)^{-\frac{1}{2}}\cos\psi \tag{5.36}$$

Eq. (5.36) was given in Vinti (1961 b). Also by (4.12) and the results

$$j_{13} = O(J_2) \qquad j_{13} + 2\pi\,\mathrm{sgn}\,\alpha_3 = O(J_2) \tag{5.37}$$

from page 189 of Vinti(1961 a), we find

$$\beta_3 = h + O(J_2), \tag{5.38}$$

so that

$$\delta\beta_3 = \delta h \tag{5.39}$$

From (5.35) and (5.36) we find

$$\delta\chi = (1 - \eta_0^2 \sin^2\psi)^{-1}[(1 - \eta_0^2)^{\frac{1}{2}} \delta\psi - \tfrac{1}{2}(1 - \eta_0^2)^{-\frac{1}{2}}\eta_0 \sin 2\psi\ \delta\eta_0] \tag{5.40}$$

Then, from (5.34), (5.39), (5.40), and (5.23), it follows that

$$\delta\phi = \delta h + (1-\eta_0^2 \sin^2\psi)^{-1} \cos I[\delta\psi - \tfrac{1}{2}(\mu p)^{-\frac{1}{2}} \sin 2\psi(\sigma_m G_m + \sigma_m J_2^{-1} \tilde{G}_m)] \quad , \tag{5.41}$$

where we have used $(1 - \eta_0^2)^{\frac{1}{2}} \operatorname{sgn}\alpha_3 = \cos I$.

It is well to note here that β_3 is just as useful an orbital element as h_0 .

6. Solution for the Perturbed Delaunay Variables

We now have to solve the canonical equations (4.13) through (4.16), the perturbing potential being

$$F_1 = -\mu r e^4 \; r^{-5} \; \sigma_4 P_4(\sin\theta) \tag{6.00}$$

in the case of the residual fourth harmonic. To obtain secular and short-periodic variations through order J_2^2 and long-periodic variations through order J_2, it will suffice to use elliptic approximations in (6.00), since $\sigma_4 \equiv J_4+J_2^2 = O(J_2^2)$. Thus in (6.00) we may put

$$\sin\theta = \eta = \eta_0 \sin\psi = \eta_0 \sin(v+g) \tag{6.01}$$

$$r = \rho = a(1-e\cos E) = a(1-e^2)(1+e\cos v)^{-1} \tag{6.02}$$

and we may use

$$\ell = E - e\sin E \tag{6.03}$$

$$d\ell = (r/a)^2(1-e^2)^{-\frac{1}{2}}dv \tag{6.04}$$

$$\psi = v + g \tag{6.05}$$

Then

$$F_1 = \frac{-\mu r_e^4 \sigma_4}{8a^5}\left[(3-15\eta_0^2+\frac{105}{8}\eta_0^4)(\frac{a}{r})^5+5\eta_0^2(3-\frac{7}{2}\eta_0^2)(\frac{a}{r})^5\cos 2\psi+\frac{35}{8}\eta_0^4(\frac{a}{r})^5\cos 4\psi\right] \tag{6.06}$$

The part of F_1 independent of ℓ is then

$$F_{1m} \equiv (2\pi)^{-1} \int_0^{2\pi} F_1 d\ell$$

$$= -\frac{\mu r_e^4 \sigma_4}{16\pi a^5}(1-e^2)^{-\frac{1}{2}} \int_0^{2\pi} [(3-15\eta_0^2+ \frac{105}{8} \eta_0^4)(\frac{a}{r})^3+5\eta_0^2(3- \frac{7}{2} \eta_0^2)(\frac{a}{r})^3\cos 2\psi + \frac{35}{8} \eta_0^4(\frac{a}{r})^3\cos 2\psi]dv \qquad (6.07)$$

From (6.02) and (6.05) it follows that

$$\int_0^{2\pi} (\frac{a}{r})^3 dv = 2\pi(1-e^2)^{-3}(1+ \frac{3}{2} e^2) \qquad (6.08)$$

$$\int_0^{2\pi} (\frac{a}{r})^3 \cos 2\psi dv = \frac{3}{2} \pi e^2(1-e^2)^{-3}\cos 2g \qquad (6.09)$$

$$\int_0^{2\pi} (\frac{a}{r})^3 \cos 4\psi dv = 0 \qquad (6.10)$$

Eqs. (6.07) through (6.10) then result in

$$F_{1m} = F_{1c} + F_{1p} , \qquad (6.11)$$

where the constant part is

$$F_{1c} = -\frac{\mu r_e^4 \sigma_4}{8a^5} (1-e^2)^{-7/2}(1+ \frac{3}{2} e^2)(3-15\eta_0^2 + \frac{105}{8} \eta_0^4) \qquad (6.12)$$

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and the long-periodic part is

$$F_{1p} = -\frac{15\mu r_e^4 \sigma_4}{32a^5} (1-e^2)^{-7/2} e^2 \eta_0^2 (3 - \tfrac{7}{2}\eta_0^2) \cos 2g \tag{6.13}$$

The short-periodic part is then given by

$$F_{1\ell} = F_1 - F_{1c} - F_{1p} , \tag{6.14}$$

so that

$$F_{1\ell} = \frac{-\mu r_e^4 \sigma_4}{8a^5} \Big\{ (3 - 15\eta_0^2 + \tfrac{105}{8}\eta_0^4)[(\tfrac{a}{r})^5 - (1-e^2)^{-7/2}(1 + \tfrac{3}{2}e^2)]$$

$$+ 5\eta_0^2(3 - \tfrac{7}{2}\eta_0^2)[(\tfrac{a}{r})^5 \cos 2\psi - \tfrac{3}{4}e^2(1-e^2)^{-7/2}\cos 2g] + \tfrac{35}{8}\eta_0^4(\tfrac{a}{r})^5 \cos 4\psi \Big\} \tag{6.15}$$

From (3.23) and (4.00) through (4.02) we now obtain

$$L = \mu(-2\alpha_1)^{-\frac{1}{2}} \qquad G = \alpha_2 \qquad H = \alpha_3 , \tag{6.16}$$

with neglect of terms of order J_2. To the same accuracy

$$a = -\tfrac{1}{2}\mu\alpha_1^{-1} \qquad \alpha_2^2 = \mu a(1-e^2) \qquad \alpha_3^2\alpha_2^{-2} = 1 - \eta_0^2 \tag{6.17}$$

Thus, with neglect of terms of order J_2,

$$a = L^2\mu^{-1} \qquad 1-e^2 = G^2L^{-2} \qquad \eta_0^2 = 1 - H^2G^{-2} , \tag{6.18}$$

as expected. When we later take derivatives of a generating function with respect to L,G, and H, we shall have to use the expressions (6.18) to replace them in the results by the elements a,e, and η_0 .

In solving the canonical equations (4.15) and (4.16) we first make a canonical transformation to new canonical variables L',G', H', ℓ', g', and h', so that the new Hamiltonian F^* shall be independent of ℓ' and h'. This first step will yield the short-periodic effects. To carry it out, introduce the generating function

$$S(L',G',H',\ell,g,h) = S_0 + S_1(L',G',H',\ell,g), \tag{6.19}$$

where

$$S_0 \equiv L'\ell + G'g + H'h \tag{6.20}$$

and where S_1 is to be of the first order in σ_4. Then follow the von Zeipel method, as applied by Brouwer (1959) to satellite orbits.

On splitting F^* into parts F_0^* and F_1^* , we then have

$$F_0(L,G,H) + F_1(L,G,H,\ell,g) = F_0^*(L',G',H') + F_1^*(L',G', H', g') \tag{6.21}$$

along with the following relations connecting S_1 and the primed and unprimed variables:

$$\begin{aligned} L &= \frac{\partial S}{\partial \ell} = L' + \frac{\partial S_1}{\partial \ell} \\ G &= \frac{\partial S}{\partial g} = G' + \frac{\partial S_1}{\partial g} \\ H &= \frac{\partial S}{\partial h} = H' \end{aligned} \tag{6.22}$$

$$\begin{aligned} \ell' &= \frac{\partial S}{\partial L'} = \ell + \frac{\partial S_1}{\partial L'} \\ g' &= \frac{\partial S}{\partial G'} = g + \frac{\partial S_1}{\partial G'} \\ h' &= \frac{\partial S}{\partial H'} = h + \frac{\partial S_1}{\partial H} \end{aligned} \tag{6.23}$$

Since the new Hamiltonian will not contain ℓ' or h', we shall have H' = H = constant and L' = constant. Insertion of (6.22) and (6.23) into (6.21) then gives a partial differential equation for S_1:

$$F_0\left(L' + \frac{\partial S_1}{\partial \ell}, G' + \frac{\partial S_1}{\partial g}, H\right) + F_1\left(L' + \frac{\partial S_1}{\partial \ell}, G' + \frac{\partial S_1}{\partial g}, H, \ell + \frac{\partial S_1}{\partial L'}, g + \frac{\partial S_1}{\partial G'}\right)$$

$$= F_0^*(L',G',H) + F_1^*\left(L',G',H,g + \frac{\partial S_1}{\partial G'}\right) \qquad (6.24)$$

Taylor expansion in the neighborhood of L',G',H,ℓ, and g, with rejection of terms beyond the first order in σ_4, then gives

$$F_0(L',G',H) + \frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial \ell} + \frac{\partial F_0}{\partial G'}\frac{\partial S_1}{\partial g} + F_1(L',G',H,\ell,g) = F_1^*(L',G',H) + F_1^*(L',G',H,g)$$

$$(6.25)$$

The zeroth order terms lead to

$$F_0(L',G',H) = F_0^*(L',G',H) \qquad (6.26)$$

and the first order terms to

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial \ell} + \frac{\partial F_0}{\partial G'}\frac{\partial S_1}{\partial G'} + F_{1c}(L',G',H) + F_{1p}(L',G',H,g) + F_{1\ell}(L',G',H,\ell,g) = F_1^*(L',G',H,g)$$

$$(6.27)$$

In writing down (6.27) we have used (6.14) to express F_1 as a sum of the terms F_{1c}, F_{1p}, and $F_{1\ell}$ and we have replaced g by g' in F_1^*, a permissible step involving an error of order σ_4^2. In (6.27) the terms independent of ℓ then yield

$$F_1^* = F_{1c}(L',G',H) + F_{1p}(L',G',H,g), \tag{6.28}$$

so that (6.26) and (6.28) together provide the new Hamiltonian F^*. The remaining terms, depending on ℓ, then yield

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial \ell} + \frac{\partial F_0}{\partial G'}\frac{\partial S_1}{\partial g} = - F_{1\ell}(L',G',H,\ell,g), \tag{6.29}$$

a partial differential equation for S_1. With use of (4.17) it becomes

$$2\pi\nu_1 \frac{\partial S_1}{\partial \ell} + 2\pi(\nu_2-\nu_1)\frac{\partial S_1}{\partial g} = F_{1\ell}(L',G',H,\ell,g) \tag{6.30}$$

Since $\nu_2-\nu_1$ is of order J_2 and since S_1 is to be of order $\sigma_4 \equiv J_4 + J_2^2$, it follows that $2\pi(\nu_2-\nu_1)\partial S_1/\partial g$ will be of order J_2^3 and is thus to be rejected. Thus

$$2\pi\nu_1 S_1 = \int F_{1\ell}(L',G',H,\ell,g)d\ell + \Phi(g) \tag{6.31}$$

To evaluate the integral in (6.31), apply (6.15), (6.02), (6.04), and (6.18). Since $F_{1\ell}$ has a factor $\sigma_4 = O(J_2^2)$, we can make a number of approximations at this point and still achieve our desired accuracy. These are: drop the primes from L' and G' in (6.31), place

$$2\pi\nu_1 = n \equiv \mu^{\frac{1}{2}} a^{-3/2} , \qquad (6.32)$$

express a,e, and η_0 by means of (6.18), drop the primes from L' and G' in calculating $\partial S_1/\partial L'$ and $\partial S_1/\partial G'$, and finally use (6.18) again to replace L,G, and H in the final formulas by a,e, and η_0 .

We obtain

$$S_1 = \sigma_4 Q_1[(1+ \frac{3}{2} e^2)(v-\ell)+(3e+ \frac{3}{4} e^3)\sin v + \frac{3}{4} e^2\sin 2v + \frac{e^3}{12} \sin 3v]$$

$$+ \sigma_4 Q_2[\frac{3}{4} e^2(v-\ell)\cos 2g+ \frac{e^3}{8} \sin(v-2g)+(\frac{3}{2}e + \frac{3}{8}e^3)\sin(v+2g)$$

$$+(\frac{1}{2}+ \frac{3}{4}e^2)\sin(2v+2g)+(\frac{e}{2}+ \frac{e^3}{8})\sin(3v+2g)+ \frac{3e^2}{16}\sin(4v+2g)+ \frac{e^3}{40}\sin(5v+2g)]$$

$$+\sigma_4 Q_3[\frac{e^3}{8} \sin(v+4g) + \frac{3}{8}e^2\sin(2v+4g) + (\frac{e}{2} + \frac{e^3}{8})\sin(3v + 4g)$$

$$+(\frac{1}{4} + \frac{3}{8}e^2)\sin(4v+4g)+(\frac{3e}{10} + \frac{3e^3}{40})\sin(5v+4g)+ \frac{e^2}{8} \sin(6v+4g)+ \frac{e^3}{56} \sin(7v+4g)]$$

$$+ \Phi(g), \qquad (6.33)$$

where $\Phi(g)$ is a constant of integration. Here

$$Q_i = \frac{-\mu r_e^4}{8na^5} (\frac{L}{G})^7 q_i = - \frac{1}{8}(\frac{r_e}{p})^4(\mu p)^{\frac{1}{2}} q_i , \qquad (6.34)$$

where

$$q_1 \equiv 3-15\eta_0^2+ \frac{105}{8} \eta_0^4 \qquad q_2\equiv 5\eta_0^2(3- \frac{7}{2} \eta_0^2) \qquad q_3\equiv \frac{35}{8} \eta_0^4 \qquad (6.35)$$

With use of (6.18) and (6.33) we find

$$\frac{\mu L^7}{na^5} = \mu^4 + O(J_2), \tag{6.36}$$

and

$$Q_1 = \frac{-\mu^4 r_e^4}{64G^7}\left(9 - 90\frac{H^2}{G^2} + 105\frac{H^4}{G^4}\right) \tag{6.37}$$

$$Q_2 = \frac{5\mu^4 r_e^4}{16G^7}\left(1 - \frac{H^2}{G^2}\right)\left(1 - 7\frac{H^2}{G^2}\right) \tag{6.38}$$

$$Q_3 = \frac{-35\mu^4 r_e^4}{64G^7}\left(1 - \frac{H^2}{G^2}\right)^2 , \tag{6.39}$$

through terms of order J_2^2.

To make S_1 purely short-periodic, one would have to choose $\Phi(g)$ in such a way that $2\pi\bar{S}_1 \equiv \int_0^{2\pi} S_1 d\ell$ would vanish. It turns out that $\Phi(g)$ would then not vanish, but would have to be a long-periodic term, just cancelling a long-periodic term of order J_2^2 arising from the rest of the expression for S_1. The later calculation of long-periodic effects, however, will be accurate only through order J_2. If we arbitrarily drop $\Phi(g)$, whose calculation would be extremely laborious, the net effect will be only to leave in the short-periodic terms of G, ℓ, g, and h some long-periodic impurities of order J_2^2, not affecting the accuracy of the calculation.

The errors of the short-periodic terms will be of order J_2^3, for two reasons. First, the short-periodic terms of the reference orbit were calculated only through terms of order J_2^2; second, the present calculation makes use of elliptic approximations, so that a variation of the form $\sigma_4 f$ has an error of order J_2 in f and thus of order J_2^3 in $\sigma_4 f$. For this second reason the secular corrections produced by the perturbing potential will also have errors of order J_2^3, even though the first omitted term in the von Zeipel equation (6.25) is of order σ_4^2 or J_2^4.

7. Short-Periodic Terms

It is now straightforward but tedious to calculate the short-periodic terms. From (6.22), (6.31), (6.32), and (6.15), we find, through terms of order J_2^2

$$\frac{\partial S_1}{\partial \ell} = L - L' = \sigma_4 L_4 , \tag{7.00}$$

where

$$L_4 = -\frac{r_e^4}{8a^5 n}\Big\{(3-15\eta_0^2+\frac{105}{8}\eta_0^4)[(\frac{a}{r})^5-(1-e^2)^{-7/2}(1+\frac{3}{2}e^2)]$$
$$+5\eta_0^2(3-\frac{7}{2}\eta_0^2)[(\frac{a}{r})^5\cos(2v+2g)-\frac{3e^2}{4}(1-e^2)^{-7/2}\cos 2g]+\frac{35}{8}\eta_0^4(\frac{a}{r})^5\cos(4v+4g)\Big\} \tag{7.01}$$

From (6.22) and (6.33), we find

$$\frac{\partial S_1}{\partial g} = G - G' = \sigma_4 G_4 , \tag{7.02}$$

where

$$G_4 = Q_2[-\frac{3}{2}e^2(v-\ell)\sin 2g - \frac{e^3}{4}\cos(v-2g)+(3e+\frac{3e^3}{4})\cos(v+2g)$$

$$+ (1+\frac{3}{2}e^2)\cos(2v+2g)+(e+\frac{e^3}{4})\cos(3v+2g)+\frac{3e^2}{8}\cos(4v+2g)+\frac{e^3}{20}\cos(5v+2g)]$$

$$+ Q_3[\tfrac{1}{2}e^3\cos(v+4g)+\frac{3}{2}e^2\cos(2v+4g)+(2e+\tfrac{1}{2}e^3)\cos(3v+4g)+(1+\frac{3}{2}e^2)\cos(4v+4g)$$

$$+ (\frac{6e}{5}+\frac{3e^3}{10})\cos(5v+4g)+\tfrac{1}{2}e^2\cos(6v+4g)+\frac{e^3}{14}\cos(7v+4g)] \qquad (7.03)$$

Since the dependence of S_1 on H is only through the Q's, the calculation of

$$h - h' = -\frac{\partial S_1}{\partial H} = \sigma_4 h_{41} \qquad (7.04)$$

is simple. First calculate the derivatives $\partial Q_i/\partial H$. From (6.37) through (6.39) and the sufficiently accurate relations $H/G = \cos I$, $H^2/G^2 = 1 - \eta_0^2$ and $G^2 = \mu p$, we find

$$\frac{\partial Q_1}{\partial H} = \frac{15}{16}(\frac{r_e}{p})^4(7\eta_0^2-4)\cos I$$

$$\frac{\partial Q_2}{\partial H} = \frac{5}{4}(\frac{r_e}{p})^4(3-7\eta_0^2)\cos I \qquad (7.05)$$

$$\frac{\partial Q_3}{\partial H} = \frac{35}{16}(\frac{r_e}{p})^4\eta_0^2\cos I$$

From (7.04), (6.33), and (7.05) we then obtain

$$h_{41} = -\frac{5}{16}\left(\frac{r_e}{p}\right)^4 \cos I\Big\{3(7\eta_0^2-4)\Big[(1+\tfrac{3}{2}e^2)(v-\ell)+(3e+\frac{3e^3}{4})\sin v+\frac{3e^2}{4}\sin 2v+\frac{e^3}{12}\sin 3v\Big]$$

$$+4(3-7\eta_0^2)\Big[\frac{3}{4}e^2(v-\ell)\cos 2g+\frac{e^3}{8}\sin(v-2g)+\left(\frac{3e}{2}+\frac{3e^3}{8}\right)\sin(v+2g)$$

$$+\left(\tfrac{1}{2}+\frac{3e^2}{4}\right)\sin(2v+2g)+\left(\frac{e}{2}+\frac{e^3}{8}\right)\sin(3v+2g)+\frac{3e^2}{16}\sin(4v+2g)+\frac{e^3}{40}\sin(5v+2g)\Big]$$

$$+7\eta_0^2\Big[\frac{e^3}{8}\sin(v+4g)+\frac{3e^2}{8}\sin(2v+4g)+\left(\frac{e}{2}+\frac{e^3}{8}\right)\sin(3v+4g)+\left(\tfrac{1}{4}+\frac{3e^2}{8}\right)\sin(4v+4g)$$

$$+\left(\frac{3e}{10}+\frac{3e^3}{40}\right)\sin(5v+4g)+\frac{e^2}{8}\sin(6v+4g)+\frac{e^3}{56}\sin(7v+4g)\Big]\Big\} \qquad (7.06)$$

To find $\ell - \ell'$ and $g - g'$, we use (6.23),(6.33), and (6.38) through (6.40). We may write (6.33) as

$$S_1 = \sigma_4 Q_1(G,H)f_1(e,v) + \sigma_4 Q_2(G,H)f_2(e,v) + \sigma_4 Q_3(G,H)f_3(e,v)\,, \qquad (7.07)$$

where

$$f_1(e,v) \equiv (1+\frac{3}{2}e^2)(v-\ell)+(3e+\frac{3e^3}{4})\sin v+\frac{3}{4}e^2\sin 2v+\frac{e^3}{12}\sin 3v \qquad (7.08)$$

$$f_2(e,v) \equiv \frac{3}{4}e^2(v-\ell)\cos 2g+\frac{e^3}{8}\sin(v-2g)+\left(\frac{3e}{2}+\frac{3e^3}{8}\right)\sin(v+2g)$$

$$+\left(\tfrac{1}{2}+\frac{3}{4}e^2\right)\sin(2v+2g)+\left(\frac{e}{2}+\frac{e^3}{8}\right)\sin(3v+2g)+\frac{3e^2}{16}\sin(4v+2g)+\frac{e^3}{40}\sin(5v+2g)$$

$$(7.09)$$

$$f_3(e,v) \equiv \frac{e^3}{8}\sin(v+4g) + \frac{3e^2}{8}\sin(2v+2g) + \left(\frac{e}{2} + \frac{e^3}{8}\right)\sin(3v+4g)$$

$$+\left(\frac{1}{4} + \frac{3e^2}{8}\right)\sin(4v+4g) + \left(\frac{3e}{10} + \frac{3e^3}{40}\right)\sin(5v+4g) + \frac{e^2}{8}\sin(6v+4g) + \frac{e^3}{56}\sin(7v+4g) \tag{7.10}$$

Then

$$\frac{\partial S_1}{\partial L} = \sigma_4 \sum_{i=1}^{3} Q_i\left(\frac{\partial f_i}{\partial e} + \frac{\partial f_i}{\partial v}\frac{\partial v}{\partial e}\right)\frac{\partial e}{\partial L} \tag{7.11}$$

$$\frac{\partial S_1}{\partial G} = \sigma_4 \sum_{i=1}^{3} \frac{\partial Q_i}{\partial G} f_i + \sigma_4 \sum_{i=1}^{3} Q_i\left(\frac{\partial f_i}{\partial e} + \frac{\partial f_i}{\partial v}\frac{\partial v}{\partial e}\right)\frac{\partial e}{\partial G}$$

$$= \sigma_4 \sum_{i=1}^{3} \frac{\partial Q_i}{\partial G} f_i + \frac{\partial S_1}{\partial L}\frac{\partial e}{\partial G}\left(\frac{\partial e}{\partial L}\right)^{-1} \tag{7.12}$$

From (6.37) through (6.39), with the same approximations used in obtaining (7.05), we find

$$\frac{\partial Q_1}{\partial G} = \frac{3}{64}\left(\frac{r_e}{p}\right)^4(136 - 500\eta_0^2 + 385\eta_0^4)$$

$$\frac{\partial Q_2}{\partial G} = -\frac{5}{16}\left(\frac{r_e}{p}\right)^4(12 - 82\eta_0^2 + 77\eta_0^4) \tag{7.13}$$

$$\frac{\partial Q_3}{\partial G} = \frac{35}{64}\left(\frac{r_e}{p}\right)^4 \eta_0^2(11\eta_0^2 - 4)$$

Next we need the elliptic approximations

$$\frac{\partial e}{\partial L} = (\mu a)^{-\frac{1}{2}} e^{-1}(1-e^2) \qquad (7.14)$$

$$\frac{\partial e}{\partial G} = -e^{-1}\left(\frac{1-e^2}{\mu a} \right)^{\frac{1}{2}} \qquad (7.15)$$

$$\frac{\partial v}{\partial e} = \left(\frac{1}{1-e^2} + \frac{a}{r}\right) \sin v = (1-e^2)^{-1}(2\sin v+\tfrac{1}{2}e\sin 2v) \qquad (7.16)$$

To compute the $\partial S_1/\partial L$ and $\partial S_1/\partial G$ we need finally the six derivatives $\partial f_i/\partial e$ and $\partial f_i/\partial v$, $i = 1,2,3$. From (7.08) through (7.10) we find

$$\frac{\partial f_1}{\partial e} = 3e(v-\ell) + \left(3+ \frac{9e^2}{4}\right)\sin v + \frac{3}{2}e\sin 2v + \frac{e^3}{4} \sin 3v \qquad (7.17)$$

$$\frac{\partial f_1}{\partial v} = 1 + \frac{3}{2}e^2 + \left(3e + \frac{3e^3}{4}\right)\cos v + \frac{3}{2}e^2\cos 2v + \frac{e^3}{4} \cos 3v \qquad (7.18)$$

$$\frac{\partial f_2}{\partial e} = \frac{3}{2}e(v-\ell)\cos 2g + \frac{3e^2}{8} \sin(v-2g)+\left(\frac{3}{2} + \frac{9e^2}{8}\right)\sin(v+2g)$$

$$+ \frac{3}{2}e \sin(2v+2g)+\left(\frac{1}{2} + \frac{3e^2}{8}\right)\sin(3v+2g)+ \frac{3e}{8} \sin(4v+2g)+ \frac{3e^2}{40}\sin(5v+2g) \qquad (7.19)$$

$$\frac{\partial f_2}{\partial v} = \frac{3}{4}e^2\cos 2g + \frac{e^3}{8} \cos(v-2g)+\left(\frac{3}{2}e+ \frac{3e^3}{8}\right)\cos(v+2g)+\left(1+\frac{3}{2}e^2\right)\cos(2v+2g)$$

$$+\left(\frac{3}{2}e + \frac{3e^3}{8}\right)\cos(3v + 2g)+ \frac{3e^2}{4}\cos(4v+2g) + \frac{e^3}{8} \cos(5v+2g) \qquad (7.20)$$

$$\frac{\partial f_3}{\partial e} = \frac{3e^2}{8}\sin(v+4g)+\frac{3e}{4}\sin(2v+4g)+(\tfrac{1}{2}+\frac{3e^2}{8})\sin(3v+4g)+\frac{3e}{4}\sin(4v+4g)$$

$$+(\frac{3}{10}+\frac{9e^2}{40})\sin(5v+4g)+\frac{e}{4}\sin(6v+4g)+\frac{3e^2}{56}\sin(7v+4g) \qquad (7.21)$$

$$\frac{\partial f_3}{\partial v} = \frac{e^3}{8}\cos(v+4g)+\frac{3e^2}{4}\cos(2v+4g)+(\frac{3e}{2}+\frac{3e^3}{8})\cos(3v+4g)+(1+\frac{3}{2}e^2)\cos(4v+4g)$$

$$+(\frac{3e}{2}+\frac{3e^3}{8})\cos(5v+4g)+\frac{3e^2}{4}\cos(6v+4g)+\frac{e^3}{8}\cos(7v+4g) \qquad (7.22)$$

There then follow

$$(1-e^2)(\frac{\partial f_1}{\partial e}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial e}) = 3e(1-e^2)(v-\ell)+(5+\frac{3e^2}{2}-\frac{17e^4}{8})\sin v+(5e-\frac{e^3}{4})\sin$$

$$+(\frac{5e^2}{2}-\frac{e^4}{16})\sin 3v+\frac{5e^3}{8}\sin 4v+\frac{e^4}{16}\sin 5v$$

$$(7.23)$$

$$(1-e^2)(\frac{\partial f_2}{\partial e}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial e}) = -\frac{e}{8}(14+5e^2)\sin 2g+\frac{3e}{2}(1-e^2)(v-\ell)\cos 2g$$

$$+(\frac{3e^2}{2}-\frac{9e^4}{32})\sin(v-2g)+\frac{5e^3}{16}\sin(2v-2g)+\frac{e^4}{32}\sin(3v-2g)$$

$$+(\tfrac{1}{2}-\frac{3}{2}e^2-\frac{19e^4}{16})\sin(v+2g)+(\frac{3}{2}e-\frac{3}{2}e^3)\sin(2v+2g)+(\frac{3}{2}+e^2-\frac{5e^4}{16})\sin(3v+2g)$$

$$+(\frac{17e}{8}+\frac{e^3}{4})\sin(4v+2g)+(\frac{6e^2}{5}+\frac{3e^4}{160})\sin(5v+2g)$$

$$+\frac{5e^3}{16}\sin(6v+2g)+\frac{e^4}{32}\sin(7v+2g) \qquad (7.24)$$

$$(1-e^2)\left(\frac{\partial f_3}{\partial e}+\frac{\partial f_3}{\partial v}\frac{\partial v}{\partial e}\right)=\frac{e^4}{32}\sin(v-4g)-\frac{5e^3}{16}\sin 4g-\left(\frac{3e^2}{4}+\frac{15e^4}{32}\right)\sin(v+4g)$$

$$-\left(e+\frac{11e^3}{8}\right)\sin(2v+4g)-\left(\frac{1}{2}+\frac{5}{4}e^2+\frac{7e^4}{16}\right)\sin(3v+4g)+\left(\frac{3e}{4}-\frac{3e^2}{4}\right)\sin(4v+4g)$$

$$+\left(\frac{13}{10}+\frac{21e^2}{20}-\frac{13e^4}{80}\right)\sin(5v+4g)+\left(2e+\frac{3e^3}{8}\right)\sin(6v+4g)$$

$$+\left(\frac{33e^2}{28}+\frac{9e^4}{224}\right)\sin(7v+4g)+\frac{5e^3}{16}\sin(8v+4g)+\frac{e^4}{32}\sin(9v+4g) \qquad (7.25)$$

From (6.34), (7.14), and (7.16) it follows that

$$Q_i\frac{\partial e}{\partial L}=-\frac{1}{8}\left(\frac{r_e}{p}\right)^4\left(\frac{1-e^2}{e}\right)^{3/2}q_i \quad (i=1,2,3) \qquad (7.26)$$

and then from (6.23), (7.11), and (7.23) through (7.26) that

$$\ell-\ell'=\sigma_4\ell_{41} \quad , \qquad (7.27)$$

where

$$\ell_{41}=\frac{(1-e^2)^{\frac{1}{2}}}{2048}\left(\frac{r_e}{p}\right)^4\Big\{6(8-40\eta_0^2+35\eta_0^4)[48(1-e^2)(v-\ell)+\frac{2}{e}(40+12e^2-17e^4)\sin v$$

$$+4(20-e^2)\sin 2v+e(40-e^2)\sin 3v+10e^2\sin 4v+e^3\sin 5v]$$

$$-4\eta_0^2(7\eta_0^2-6)[240(1-e^2)(v-\ell)\cos 2g+5e^3\sin(3v-2g)+50e^2\sin(2v-2g)$$

$$+ 15e(16-3e^2)\sin(v-2g)-20(14+5e^2)\sin 2g+ \frac{10}{e}(8-24e^2-19e^4)\sin(v+2g)$$

$$+ 240(1-e^2)\sin(2v+2g)+ \frac{10}{e}(24+16e^2-5e^4)\sin(3v+2g)+20(17+2e^2)\sin(4v+2g)$$

$$+ 3e(64+e^2)\sin(5v+2g)+50e^2\sin(6v+2g)+5e^3\sin(7v+2g)]$$

$$+ \eta_0^4[35e^3\sin(v-4g)-350e^2\sin 4g-105e(8+5e^2)\sin(v+4g)$$

$$- 140(8+11e^2)\sin(2v+4g)- \frac{70}{e}(8+20e^2+7e^4)\sin(3v+4g)+840(1-e^2)\sin(4v+4g)$$

$$+ \frac{14}{e}(104+84e^2-13e^4)\sin(5v+4g)+140(16+3e^2)\sin(6v+4g)$$

$$+15e(88+3e^2)\sin(7v+4g)+350e^2\sin(8v+4g)+35e^3\sin(9v+4g)]\Big\} \qquad (7.28)$$

From (6.23), (7.12), (7.14), and (7.15), we find

$$g-g' = -(1-e^2)^{-\frac{1}{2}}(\ell-\ell') - \sigma_4 \sum_{i=1}^{3} \frac{\partial Q_1}{\partial G_i} f_i \qquad (7.29)$$

Then, from (7.13) and (7.08) through (7.10), we find

$$g-g' = \sigma_4 g_{41} , \qquad (7.30)$$

where

$$g_{41} = -(1-e^2)^{-\frac{1}{2}}\ell_{41}- \frac{1}{512}\left(\frac{r_e}{p}\right)^4\Big\{2(136-500\eta_0^2+ 385\eta_0^4)[6(2+3e^2)(v-\ell)$$

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$$9e(4+e^2)\sin v+9e^2\sin 2v+e^3\sin 3v]-2(12-82\eta_0^2+11\eta_0^4)[60e^2(v-\ell)\cos 2g$$

$$+ 10e^3\sin(v-2g)+30e(4+e^2)\sin(v+2g)+20(2+3e^2)\sin(2v+2g)$$

$$+ 10e(4+e^2)\sin(3v+2g)+15e^2\sin(4v+2g)+2e^3\sin(5v+2g)]$$

$$+ \eta_0^2(11\eta_0^2-4)[35e^3\sin(v+4g)+105e^2\sin(2v+4g)+35e(4+e^2)\sin(3v+4g)$$

$$+ 35(2+3e^2)\sin(4v+4g)+21e(4+3e^2)\sin(5v+4g)$$

$$+ 35e^2\sin(6v+4g) + 5e^3\sin(7v+4g)]\} \qquad (7.31)$$

This concludes the solution for the short-periodic terms arising from the residual fourth harmonic, with coefficient $\sigma_4 \equiv J_4+J_2^2$.

8. Long-Periodic Terms

By (6.21) and (6.28) the Hamiltonian F^* is

$$F^* = F_0(L',G',H) + F_{1c}(L',G',H) + F_{1p}(L',G',H,g') , \qquad (8.00)$$

short-periodic terms having been eliminated. We now try to find new canonical variables L'',G'',H'',ℓ'',g'', and h'', corresponding to a new Hamiltonian $F_0^{**}(L'',G'',H'') + F_1^*(L'',G'',H'')$, so that

$$F_0(L',G',H) + F_{1c}(L',G',H) + F_{1p}(L',G',H,g') = F_0^{**}(L'',G'',H'')+F_1^{**}(L'',G'',H'') \qquad (8.01)$$

If we can do so, then L'', G'', and H'' will be constants of the motion and

$$\dot{\ell}'' = -\frac{\partial F^{**}}{\partial L''} \qquad \dot{g}'' = -\frac{\partial F^{**}}{\partial G''} \qquad \dot{h}'' = -\frac{\partial F^{**}}{\partial H''} \tag{8.02}$$

To find the necessary canonical transformation, we introduce the generating function

$$S^{*} = L''\ell' + G''g' + H''h' + S_1^{*}(L'',G'',H'',g') \tag{8.03}$$

where S_1^{*} is to be of order σ_4. Then

$$L' = \frac{\partial S^{*}}{\partial \ell'} = L'' \qquad\qquad \ell'' = \frac{\partial S^{*}}{\partial L''} = \ell' + \frac{\partial S_1^{*}}{\partial L''}$$

$$G' = \frac{\partial S^{*}}{\partial g'} = G'' + \frac{\partial S_1^{*}}{\partial g'} \quad (8.04) \qquad\qquad g'' = \frac{\partial S^{*}}{\partial G''} = g' + \frac{\partial S_1^{*}}{\partial G''} \quad (8.05)$$

$$H = H' = \frac{\partial S^{*}}{\partial h'} = H'' \qquad\qquad h'' = \frac{\partial S^{*}}{\partial H''} = h' + \frac{\partial S_1^{*}}{\partial H}$$

Insertion of (8.04) and (8.05) into (8.01) leads to

$$F_0^{*}\left(L',G''+\frac{\partial S_1^{*}}{\partial g'},H\right) + F_{1c}\left(L',G''+\frac{\partial S_1^{*}}{\partial g'},H\right) + F_{1p}\left(L',G''+\frac{\partial S_1^{*}}{\partial g'},H,g'+\frac{\partial S_1^{*}}{\partial G''}\right)$$

$$= F_0^{**} + F_1^{**}\,, \tag{8.06}$$

whose Taylor expansion, with neglect of terms of order σ_4^2 or higher, is

$$F_0(L',G'',H) + \frac{\partial F_0}{\partial G''}\frac{\partial S_1^*}{\partial g'} + F_{1c}(L',G'',H) + F_{1p}(L',G'',H,g')$$

$$= F_0^{**} + F_1^{**} \tag{8.07}$$

Splitting (8.07) into zeroth order and first order terms yields, respectively,

$$F_0^{**} = F_0(L',G'',H) \tag{8.08}$$

$$\frac{\partial F_0}{\partial G''}\ \frac{\partial S_1^*}{\partial g'} + F_{1c} + F_{1p} = F_1^{**} \tag{8.09}$$

Resolution of (8.09) into constant and long-periodic terms then shows that

$$F_1^{**} = F_{1c}(L',G'',H) \tag{8.10}$$

$$\frac{\partial F_0}{\partial G''}\frac{\partial S_1^*}{\partial g'} = - F_{1p}(L',G'',H,g') \tag{8.11}$$

With use of (4.17) and (6.13) and of double primes to denote quantities corresponding to G", (8.11) becomes

$$2\pi(\nu_1''- \nu_2'')\frac{\partial S_1^*}{\partial g'} = \frac{15\mu r_e^4\sigma_4}{32a'^5}\, e''^2(1-e''^2)^{-7/2}\eta_0''^2(3 - \frac{7}{2}\,\eta_0''^2)\cos 2g' \tag{8.12}$$

By Eqs. (7.34) and (7.37) of an earlier paper (Vinti 1961 a), we have

$$2\pi(\nu_1''-\nu_2'') = -\frac{3r_e^2J_2n}{4p^2}(5\cos^2 I - 1) + O(J_2^2)$$

$$= \frac{3}{4}\,\frac{\mu^4r_e^2J_2}{L'^3G''^4}\left(1 - \frac{5H^2}{G''^2}\right) + O(J_2^2) \tag{8.13}$$

On inserting (8.13) into (8.12) and replacing elements by Delaunay variables, we find

$$\frac{\partial S_1^*}{\partial g'} = -\frac{5\mu^2r_e^2\sigma_4}{16J_2G''^3}\left(1-\frac{G''^2}{L'^2}\right)\left(1-\frac{H^2}{G''^2}\right)\left(1-7\frac{H^2}{G''^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}\cos 2g' \tag{8.14}$$

Integration then yields

$$S_1^* = -\frac{5\mu^2r_e^2\sigma_4}{32J_2G''^3}\left(1-\frac{G''^2}{L'^2}\right)\left(1-\frac{H^2}{G''^2}\right)\left(1-7\frac{H^2}{G''^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}\sin 2g' \tag{8.15}$$

Then

$$\frac{\partial S_1^*}{\partial L'} = -\frac{5\mu^2r_e^2\sigma_4}{16J_2G''L'^3}\left(1-\frac{H^2}{G''^2}\right)\left(1-7\frac{H^2}{G''^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}\sin 2g' \tag{8.16}$$

$$\frac{\partial S_1^*}{\partial G'} = \frac{5\mu^2r_e^2\sigma_4}{32J_2G''^4}\Big[3\left(1-5\frac{H^2}{G''^2}\right)-\frac{G''^2}{L'^2}\left(1-9\frac{H^2}{G''^2}\right)-\frac{8H^4}{G''^4}\left(7-5\frac{G''^2}{L'^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}$$

$$-80\frac{H^6}{G''^6}\left(1-\frac{G''^2}{L'^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-2}\Big]\sin 2g' \tag{8.17}$$

$$\frac{\partial S_1^*}{\partial H} = \frac{5\mu^2 r_e^2 \sigma_4}{16\, J_2} \frac{H}{G''^5}\left(1-\frac{G''^2}{L'^2}\right)\left[3 + 16\frac{H^2}{G''^2}\left(1-5\frac{H^2}{G''^2}\right)^{-1} + 40\frac{H^4}{G''^4}\left(1-5\frac{H^2}{G''^2}\right)^{-2}\right]\sin 2g' \tag{8.18}$$

We now express the long-periodic terms in the notation of (5.00) and (5.01).

$$G' - G'' = \frac{\sigma_4}{J_2}\tilde{G}_4 \qquad g'-g'' = \frac{\sigma_4}{J_2}\tilde{g}_4$$

$$\ell' - \ell'' = \frac{\sigma_4}{J_2}\tilde{\ell}_4 \qquad h'-h'' = \frac{\sigma_4}{J_2}\tilde{h}_4 \tag{8.19}$$

If we then use (8.04),(8.05),(8.14), and (8.16) through (8.19) and place $G''^2 = \mu p$, $L'^2 = \mu a$, $H/G'' = \cos I$, $G''^2/L'^2 = 1-e^2$, g' = unperturbed g as given by (4.18), we find to the required accuracy

$$\tilde{G}_4 = -\frac{5r_e^2 n}{16} e^2(1-e^2)^{-3/2}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\cos 2g \tag{8.20}$$

$$\tilde{\ell}_4 = \frac{5}{16}\left(\frac{r_e}{a}\right)^2(1-e^2)^{-\frac{1}{2}}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\sin 2g \tag{8.21}$$

$$\tilde{g}_4 = -\frac{5}{32}\left(\frac{r_e}{p}\right)^2[2+e^2-3(2+3e^2)\cos^2 I-8(2+5e^2)\cos^4 I(1-5\cos^2 I)^{-1}$$

$$-80e^2\cos^6 I(1-5\cos^2 I)^{-2}]\sin 2g \tag{8.22}$$

$$\tilde{h}_4 = -\frac{5}{16}\left(\frac{r_e}{p}\right)^2 e^2\cos I[3+16\cos^2 I(1-\cos^2 I)^{-1}+40\cos^4 I(1-5\cos^2 I)^{-2}]\sin 2g \tag{8.23}$$

9. Secular Effects

We now have to use (8.02),(8.08),(8.10), and (4.17) to obtain ℓ'', g'', and h'' as linear functions of the time. From (4.17) and (8.08) we obtain

$$\frac{\partial F_0^{**}}{\partial L'} = -2\pi\nu_1'' \qquad \frac{\partial F_0^{**}}{\partial G''} = 2\pi(\nu_1''-\nu_2'') \qquad \frac{\partial F_0^{**}}{\partial H} = 2\pi(\nu_2'' \operatorname{sgn}\alpha_3 - \nu_3'') \tag{9.00}$$

From (8.10) and (6.12), with use of $e'^2 = 1-G''^2/L'^2$, $L'^2 = \mu a'$, and $\eta''^2_0 = 1-H^2/G''^2$, we have with sufficient accuracy

$$F_1^{**} = -\frac{3\mu^6 r_e^4\sigma_4}{128L'^3G''^7}\left(5-3\frac{G''^2}{L'^2}\right)\left(3-30\frac{H^2}{G''^2}+35\frac{H^4}{G''^4}\right) \tag{9.01}$$

Then

$$\frac{\partial F_1^{**}}{\partial L'} = \frac{45\mu^6 r_e^4\sigma_4}{128L'^4G''^7}\left(1-\frac{G''^2}{L'^2}\right)\left(3-30\frac{H^2}{G''^2}+35\frac{H^4}{G''^4}\right) \tag{9.02}$$

$$\frac{\partial F_1^{**}}{\partial G''} = \frac{15\mu^6 r_e^4\sigma_4}{128L'^3G''^8}\left[3\left(7-3\frac{G''^2}{L'^2}\right)+18\left(7\frac{G''^2}{L'^2}-15\right)\frac{H^2}{G''^2}\right.$$

$$\left.+\ 7\left(55-27\frac{G''^2}{L'^2}\right)\frac{H^4}{G''^4}\right] \tag{9.03}$$

$$\frac{\partial F_1^{**}}{\partial H} = \frac{15\mu^6 r_e^4\sigma_4}{32L'^3G''^9}H\left(5-3\frac{G''^2}{L'^2}\right)\left(3-7\frac{H^2}{G''^2}\right) \tag{9.04}$$

Since our ~~unperturbed~~ constant orbital elements a, e, and I, and thus the frequencies ν_1 and ν_2, correspond to L', G", and H, we may drop the double primes from the ν's in (9.00). Also, to our specified accuracy, we may put $e^2 = 1-G''^2/L'^2$, $L'^2 = \mu a$, and $\eta_0^2 = 1-H^2/G''^2$ in (9.02) through (9.04). Then, with the unperturbed values ℓ, g, and h given by (4.18), we find

$$\ell'' = \ell + \sigma_4 \ell_{42} \qquad g'' = g + \sigma_4 g_{42} \qquad h'' = h + \sigma_4 h_{42}, \tag{9.05}$$

where

$$\ell_{42} = -\frac{45}{128}\left(\frac{r_e}{p}\right)^4 n(1-e^2)^{\frac{1}{2}}(3 - 30\cos^2 I + 35\cos^4 I)t \tag{9.06}$$

$$g_{42} = -\frac{15}{128}\left(\frac{r_e}{p}\right)^4 n[3(4+3e^2) - 18(8+7e^2)\cos^2 I + 7(28+27e^2)\cos^4 I]t \tag{9.07}$$

$$h_{42} = -\frac{15}{32}\left(\frac{r_e}{p}\right)^4 n(2+3e^2)\cos I(3-7\cos^2 I)t \tag{9.08}$$

We postpone summarizing the results for the residual fourth harmonic until the complete algorithm in Section 11.

10. Effects of the Third Zonal Harmonic

By (2.01) the contribution of the third zonal harmonic to the potential is

$$\Delta V = \mu\, r_e^3\, r^{-4}\, J_3 P_3(\sin\theta) \tag{10.00}$$

Corresponding to (3.02), this leads to

$$\Delta F_1 = -\mu\, r_e^3\, r^{-4}\, J_3 P_3(\sin\theta) \tag{10.01}$$

and thus to

$$\Delta F_1 = -\frac{\mu r_e^3}{a^4} J_3[-\tfrac{3}{2}\eta_0 + \tfrac{15}{8}\eta_0^3)(\tfrac{a}{r})^4 \sin(v+g) - \tfrac{5}{8}\eta_0^3(\tfrac{a}{r})^4 \sin(3v+3g)], \tag{10.02}$$

corresponding to (6.06). Following the notation of Section 6, we then find

$$\Delta F_{1m} = (2\pi)^{-1} \int_0^{2\pi} \Delta F_1 d\ell = -\frac{\mu r_e^3}{a^4} J_3 e(1-e^2)^{-5/2}(-\tfrac{3}{2}\eta_0 + \tfrac{15}{8}\eta_0^3)\sin g, \tag{10.03}$$

which is purely long-periodic, so that

$$\Delta F_{1m} = \Delta F_{1p} \quad (10.04) \qquad \Delta F_{1c} = 0 \quad (10.05)$$

Since $\Delta F_{1c} = 0$, there are no secular effects.

The short-periodic part of ΔF_1 is then

$$\Delta F_{1\ell} = \Delta F_1 - \Delta F_{1m} = -\frac{\mu r_e^3}{a^4} J_3[(-\tfrac{3}{2}\eta_0 + \tfrac{15}{8}\eta_0^3)\{(\tfrac{a}{r})^4 \sin(v+g)$$
$$- e(1-e^2)^{-5/2} \sin g\} - \tfrac{5}{8}\eta_0^3(\tfrac{a}{r})^4 \sin(3v+3g)] \tag{10.06}$$

On following the same procedure as in Section 6, we find

$$\frac{\partial \Delta S_1}{\partial \ell} = -(2\pi\nu_1)^{-1} \Delta F_{1\ell} \tag{10.07}$$

where

$$\Delta S_1 = -(2\pi\nu_1)^{-1} \int \Delta F_{1\ell}\, d\ell \tag{10.08}$$

$$= -\frac{\mu r_e^3}{96na^4} J_3(1-e^2)^{-5/2}\, \eta_0 \Big\{3(1-5\cos^2 I)[12e(v-\ell)\sin g + 3e^2\cos(v-g)$$

$$-6(2+e^2)\cos(v+g) - 6e\cos(2v+g) - e^2\cos(3v+g)] + \eta_0^2[15e^2\cos(v+3g) + 30e\cos(2v+3g)$$

$$+ 10(2+e^2)\cos(3v+3g) + 15e\cos(4v+3g) + 3e^2\cos(5v+3g)]\Big\} \tag{10.09}$$

Before taking derivatives of (10.09) with respect to L' and G' one must replace e^2 by $1-G'^2/L'^2$, η_0^2 by $1-H^2/G'^2$, and a by L'^2/μ. The short-periodic effects are then given by

$$\frac{\partial \Delta S_1}{\partial \ell} = L - L' = J_3 L_3 \qquad\qquad -\frac{\partial \Delta S_1}{\partial L'} = \ell - \ell' = J_3 \ell_3$$

$$\frac{\partial \Delta S_1}{\partial g} = G - G' = J_3 G_3 \quad (10.10) \qquad\qquad -\frac{\partial \Delta S_1}{\partial G'} = g - g' = J_3 g_3 \quad (10.11)$$

$$\frac{\partial \Delta S_1}{\partial h} = H - H' = 0 \qquad\qquad -\frac{\partial \Delta S_1}{\partial H} = h - h' = J_3 h_3 \,,$$

where

$$L_3 = -\frac{nr_e^3}{a}\left\{\left(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3\right)\left[\left(\frac{a}{r}\right)^4\sin(v+g) - e(1-e^2)^{-5/2}\sin g\right]\right.$$

$$\left. - \frac{5}{8}\eta_0^3\left(\frac{a}{r}\right)^4\sin(3v+3g)]\right\} \qquad (10.12)$$

$$G_3 = -na^3(1-e^2)^{\frac{1}{2}}\left(\frac{r_e}{p}\right)^3\left\{\left(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3\right)\left[e(v-\ell)\cos g + \frac{e^2}{4}\sin(v-g)\right.\right.$$

$$+(1+\tfrac{1}{2}e^2)\sin(v+g) + \tfrac{1}{2}e\sin(2v+g) + \frac{e^2}{12}\sin(3v+g)\Big] - \frac{15}{8}\eta_0^3\Big[\frac{e^2}{4}\sin(v+3g) + \tfrac{1}{2}e\sin(2v+3g)$$

$$\left. + \frac{1}{3}(1+\tfrac{1}{2}e^2)\sin(3v+3g) + \frac{e}{4}\sin(4v+3g) + \frac{e^2}{20}\sin(5v+3g)\Big]\right\} \qquad (10.13)$$

$$\ell_3 = \frac{\eta_0}{8e}(1-e^2)^{\frac{1}{2}}\left(\frac{r_e}{p}\right)^3\left\{3(5\eta_0^2-4)\left[(1-e^2)(v-\ell)\sin g + (1+\tfrac{1}{2}e^2)\cos g\right.\right.$$

$$+ \frac{e}{8}(14-3e^2)\cos(v-g) + \tfrac{1}{2}e^2\cos(2v-g) + \frac{e^3}{16}\cos(3v-g) - e(1-e^2)\cos(v+g)$$

$$- \frac{3}{2}\cos(2v+g) - \frac{e}{24}(34-e^2)\cos(3v+g) - \tfrac{1}{2}e^2\cos(4v+g) - \frac{e^3}{16}\cos(5v+g)\Big]$$

$$+ 5\eta_0^2\Big[-\tfrac{1}{2}e^2\cos 3g - \frac{e^3}{16}\cos(v-3g) - \frac{e}{8}(6+5e^2)\cos(v+3g) - (\tfrac{1}{2}+e^2)\cos(2v+3g)$$

$$+ \frac{e}{3}(1-e^2)\cos(3v+3g) + \tfrac{1}{4}(5+e^2)\cos(4v+3g) + \frac{e}{40}(54+e^2)\cos(5v+3g)$$

$$\left. + \tfrac{1}{2}e^2\cos(6v+3g) + \frac{e^3}{16}\cos(7v+3g)\Big]\right\} \qquad (10.14)$$

$$g_3 = -(1-e^2)^{-\frac{1}{2}}\ell_3 - \frac{1}{8}\left(\frac{r_e}{p}\right)^3\left\{\eta_0^{-1}(4-39\eta_0^2+40\eta_0^4)\left[e(v-\ell)\sin g + \frac{e^4}{4}\cos(v-g)\right.\right.$$

$$-(1+\tfrac{1}{2}e^2)\cos(v+g) - \tfrac{1}{2}e\cos(2v+g) - \frac{e^2}{12}\cos(3v+g)\Big] + 5\eta_0(8\eta_0^2-3)\Big[\frac{e^2}{4}\cos(v+3g)$$

$$+\tfrac{1}{2}e\cos(2v+3g) + \tfrac{1}{3}(1+\tfrac{1}{2}e^2)\cos(3v+3g) + \tfrac{e}{4}\cos(4v+3g) + \tfrac{e^2}{20}\cos(5v+3g)]\} \qquad (10.15)$$

$$h_3 = \left(\frac{r_e}{p}\right)^3 \{ \tfrac{3}{2}\cot I(1-\tfrac{15}{4}\sin^2 I)[e(v-\ell)\sin g + \tfrac{e^2}{4}\cos(v-g) - (1+\tfrac{1}{2}e^2)\cos(v+g)$$

$$-\tfrac{1}{2}e\cos(2v+2g) - \tfrac{e^2}{12}\cos(3v+2g)] - \tfrac{15}{16}\sin 2I[\tfrac{e^2}{4}\cos(v+3g)$$

$$+\tfrac{1}{2}e\cos(2v+3g) + \tfrac{1}{3}(1+\tfrac{1}{2}e^2)\cos(3v+3g) + \tfrac{e}{4}\cos(4v+3g) + \tfrac{e^2}{20}\cos(5v+3g)]\} \qquad (10.16)$$

Continuing on to the long-periodic effects and following the procedures of Section 8, we find

$$(\Delta F_1)^{**} = 0 , \qquad (10.17)$$

so that the third harmonic gives no secular changes, and

$$2\pi(\nu_1'' - \nu_2'') \frac{\partial \Delta S_1^*}{\partial g'} = -\Delta F_{1p} \qquad (10.18)$$

Now

$$F_{1p} = -\frac{3}{8}\frac{\mu^5 r_e^3 e''}{L'^3 G''^5} J_3 \eta_0''(1-5\frac{H^2}{G''^2})\sin g', \qquad (10.19)$$

by (10.03) and (10.04). With use of (10.18) and (10.19) and of (8.13) for $2\pi(\nu_1'' - \nu_2'')$, we find

$$\frac{\partial \Delta S_1^*}{\partial g'} = \frac{1}{2}\frac{\mu r_e}{G''}\frac{J_3}{J_2} e'' \eta_0'' \sin g' , \qquad (10.20)$$

so that

$$\Delta S_1^* = -\tfrac{1}{2}\frac{\mu r_e}{G''}\frac{J_3}{J_2}e''\eta_0''\cos g' \tag{10.21}$$

It follows that

$$\frac{\partial\Delta S_1^*}{\partial \ell'} = L'-L'' = 0 \qquad\qquad -\frac{\partial\Delta S_1^*}{\partial L'} = \ell'-\ell'' = J_3\tilde{\ell}_3$$

$$\frac{\partial\Delta S_1^*}{\partial g'} = G'-G'' = J_3\tilde{G}_3 \quad (10.22) \qquad\qquad -\frac{\partial\Delta S_1^*}{\partial G'} = g'-g'' = J_3\tilde{g}_3 \quad (10.23)$$

$$\frac{\partial\Delta S_1^*}{\partial h'} = H'-H'' = 0 \qquad\qquad -\frac{\partial\Delta S_1^*}{\partial H'} = h'-h'' = J_3\tilde{h}_3 \quad ,$$

where

$$\tilde{G}_3 = \tfrac{1}{2}\, r_e na(1-e^2)^{-\frac{1}{2}} e\eta_0 \sin g \tag{10.24}$$

$$\tilde{\ell}_3 = \tfrac{1}{2}\frac{r_e}{a}\frac{(1-e^2)^{\frac{1}{2}}}{e}\sin I\,\cos g \tag{10.25}$$

$$\tilde{g}_3 = \tfrac{1}{2}\frac{r_e}{p}\left(\frac{e\cos^2 I}{\sin I} - \frac{\sin I}{e}\right)\cos g \tag{10.26}$$

$$\tilde{h}_3 = -\tfrac{1}{2}\frac{r_e}{p}\, e\cot I\,\cos g \tag{10.27}$$

A summary of the results for the third harmonic will appear in the next section, which gives the complete algorithm.

11. The Complete Algorithm

We shall summarize results by writing out the complete algorithm for the calculation of the motion in the potential field (1.00), as modified by the third harmonic and the residual fourth harmonic. This will involve a repetition, with some changes, of about two pages of an earlier paper (Vinti 1961 a), but it is highly desirable to assemble the whole solution in one place.

Let the planetary constants μ, r_e, J_2, J_3, and J_4 be given, along with the constant orbital elements a, e, I, ℓ_0, g_0, and β_3. To calculate the

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compute

$$c = r_e J_2^{\frac{1}{2}} \qquad \eta_0 = \sin I \qquad p = a(1-e^2)$$

$$D = (ap-c^2)(ap-c^2\eta_0^2) + 4a^2c^2\eta_0^2 \qquad D' = 4a^2c^2(1-\eta_0^2) + D$$

$$A = -2ac^2D^{-1}(1-\eta_0^2)(ap-c^2\eta_0^2) \qquad B = c^2\eta_0^2D^{-1}D'$$

$$b_1 = -\tfrac{1}{2}A \qquad b_2 = B^{\frac{1}{2}} \qquad -2\alpha_1 = \mu(a+b_1)^{-1}$$

$$-\frac{\alpha_2^2}{2\alpha_1} = a_0p_0 = -c^2(1-\eta_0^2) + apD^{-1}D' \qquad \alpha_2 = (-2\alpha_1)^{\frac{1}{2}}(a_0p_0)^{\frac{1}{2}} > 0$$

$$\alpha_3 = \alpha_2\left(1-\frac{c^2\eta_0^2}{a_0p_0}\right)^{\frac{1}{2}}\cos I \qquad \eta_2^{-2} = \frac{c^2D}{apD'} \qquad k = c^2p^{-2} \qquad q = \eta_0\eta_2^{-1}$$

$$A_1 = (1-e^2)^{\frac{1}{2}} p \sum_{n=2}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_{n-2}[(1-e^2)^{\frac{1}{2}}]$$

$$A_2 = (1-e^2)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_n[(1-e^2)^{\frac{1}{2}}] ,$$

where $P_n(x)$ is the Legendre polynomial of degree n and where $R_n(x) \equiv x^n P_n(x^{-1})$, always a polynomial of degree $[n/2]$ in x^2 .

$$A_3 = (1-e^2)^{\frac{1}{2}} p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2} [(1-e^2)^{\frac{1}{2}}],$$

where

$$D_{2i} = \sum_{n=0}^{i} (-1)^{i-n} (c/p)^{2i-2n} (b_2/p)^{2n} P_{2n}(b_1/b_2)$$

$$D_{2i+1} = \sum_{n=0}^{i} (-1)^{i-n} (c/p)^{2i-2n} (b_2/p)^{2n+1} P_{2n+1}(b_1/b_2)$$

$$B_1 = 2\pi^{-1} q^{-2}[K(q) - E(q)] \qquad B_2 = 2\pi^{-1} K(q) ,$$

where $K(q)$ and $E(q)$ are the complete elliptic integrals of the first and second kinds, respectively. It may be convenient to have the series

$$B_1 = \tfrac{1}{2} + \tfrac{3}{16} q^2 + \tfrac{15}{128} q^4 + \ldots \qquad B_2 = 1 + \tfrac{1}{4} q^2 + \tfrac{9}{64} q^4 + \ldots$$

Continue with

$$B_3 = 1 - (1-\eta_2^{-2})^{-\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m} \text{, where } \gamma_m = \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{(2n)!\ \eta_0^{2n}}{2^{2n}(n!)^2}$$

$$A_{11} = \frac{3}{4}(1-e^2)^{\frac{1}{2}} p^{-3} e(-2b_1 b_2^2 p + b_2^4) \qquad A_{12} = \frac{3}{32}(1-e^2)^{\frac{1}{2}} p^{-3} b_2^4 e^2$$

$$A_{21} = (1-e^2)^{\frac{1}{2}} p^{-1} e[b_1 p^{-1} + (3b_1^2 - b_2^2)p^{-2} - \frac{9}{2} b_1 b_2^2 (1+\frac{e^2}{4})p^{-3} + \frac{3}{8} b_2^4 (4+3e^2)p^{-4}]$$

$$A_{22} = (1-e^2)^{\frac{1}{2}} p^{-1} [\frac{e^2}{8} (3b_1^2 - b_2^2)p^{-2} - \frac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3}{32} b_2^4 (6e^2 + e^4)p^{-4}]$$

$$A_{23} = (1-e^2)^{\frac{1}{2}} p^{-1} \frac{e^3}{8}(-b_1 b_2^2 p^{-3} + b_2^4 p^{-4})$$

$$A_{24} = \frac{3}{256}(1-e^2)^{\frac{1}{2}} p^{-5} b_2^4 e^4$$

$$A_{31} = (1-e^2)^{\frac{1}{2}} p^{-3} e[2 + b_1 p^{-1}(3 + \frac{3e^2}{4}) - p^{-2}(\frac{1}{2} b_2^2 + c^2)(4 + 3e^2)]$$

$$A_{32} = (1-e^2)^{\frac{1}{2}} p^{-3} [\frac{e^2}{4} + \frac{3}{4} b_1 p^{-1} e^2 - p^{-2}(\frac{e^2}{4} + \frac{3}{2} e^2)(\frac{1}{2} b_2^2 + c^2)]$$

$$A_{33} = (1-e^2)^{\frac{1}{2}} p^{-3} e^3 [\frac{b_1}{12} p^{-1} - \frac{1}{3} p^{-2} (\frac{1}{2} b_2^2 + c^2)]$$

$$A_{34} = -\frac{1}{32} (1-e^2)^{\frac{1}{2}} p^{-5} e^4 (\frac{1}{2} b_2^2 + c^2)$$

$$2\pi\nu_1 = (-2\alpha_1)^{\frac{1}{2}} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1}$$

$$2\pi\nu_2 = (\alpha_2^2 - \alpha_3^2)^{\frac{1}{2}} \eta_0^{-1} A_2 B_2^{-1} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1}$$

where

$$M_1 = (a+b_1)^{-1}[-(A_1+c^2\eta_0^2\, A_2B_1B_2^{-1})v_0+ \frac{c^2}{4}(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\, \sin(2\psi_s+2\psi_0)]$$

The term v_1 is then given by placing $v = M_s+v_0+v_1$ and $E = M_s+E_0+E_1$ in the anomaly connections. Then

$$\psi_1 = (-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_1 + A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s + 2v_0)] + \frac{q^2}{8} B_2^{-1} \sin(2\psi_s + 2\psi_0)$$

Finally

$$E_2 = [1-e'\cos(M_s+E_0+E_1)]^{-1}M_2,$$

where

$$M_2 = -(a+b_1)^{-1}[A_1v_1+A_{11} \sin(M_s+v_0)+ A_{12} \sin(2M_s + 2v_0)$$
$$+ c^2(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\{B_1\psi_1-\tfrac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)- \frac{q^2}{8} \sin(2\psi_s+2\psi_0)+ \frac{q^2}{64}\sin(4\psi_s+4\psi_0)\}$$

Then v_2 is found by placing $v = M_s+v_0+v_1+v_2$ and $E= M_s+E_0+E_1+E_2$ in the anomaly connections and

$$\psi_2=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\, \eta_0^{-1}\, B_2^{-1}[A_2v_2 + A_{21}v_1 \cos(M_s+v_0)+ 2A_{22}v_1 \cos(2M_s+2v_0)$$
$$+ A_{23}\sin(3M_s+3v_0)+ A_{24}\sin(4M_s+4v_0)]+ \frac{q^2}{4} B_2^{-1}[\psi_1\cos(2\psi_s + 2\psi_0)$$
$$+ \frac{3q^2}{8} \sin(2\psi_s+ 2\psi_0) - \frac{3q^2}{64} \sin(4\psi_s+ 4\psi_0)]$$

The uniformising variables E, v, and ψ are then given by $E = M_s+E_p$, $v = M_s+E_p$, and $\psi = \psi_s+\psi_p$. If t is the time, their secular parts M_s and ψ_s are given exactly by

$$M_s = \ell_0 + 2\pi\nu_1 t \qquad \psi_s = \ell_0 + g_0 + 2\pi\nu_2 t$$

Let the periodic parts be split as follows: $E_p \doteq E_0+E_1+E_2$, $v_p = v_0 + v_1 + v_2$, and $\psi_p = \psi_0 + \psi_1 + \psi_2$, where, e.g., E_0 contains terms of order J_2^0, J_2, and J_2^2, E_1 contains terms of order J_2 and J_2^2, and E_2 contains terms of order J_2^2 only.

Then E_0 is given by the Kepler equation

$$M_s + E_0 - e'\sin(M_s+E_0) = M_s \;,$$

where $e' \equiv ae(a+b_1)^{-1} < e$. The term v_0 is then given by placing $v = M_s+v_0$ and $E = M_s+v_0$ in the anomaly connections

$$\cos v = (\cos E - e)(1-e\cos E)^{-1} \qquad \sin v = (1-e^2)^{\frac{1}{2}}(1-e\cos E)^{-1}\sin E$$

or equivalent relations. (Note that e here is the original e and not the e' in the Kepler equation.) Then

$$\psi_0 = (-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2 - \alpha_3^2)^{\frac{1}{2}}\, \eta_0^{-1}\, A_2\, B_2^{-1}\, v_0$$

The term E_1 is now given by

$$E_1 = [1-e'\cos(M_s+E_0)]^{-1}\, M_1 - \tfrac{1}{2}e'[1-e'\cos(M_s+E_0)]^{-3}\, M_1^2\, \sin(M_s+E_0) \;,$$

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The spheroidal coordinates ρ and η are then given by

$$\rho = a(1-e\cos E) = (1+e\cos v)^{-1}p \qquad \eta = \eta_0 \sin\psi ,$$

where $E = M_s+E_0+E_1+E_2$ and $v = M_s+v_0+v_1+v_2$. To obtain the unperturbed right ascension ϕ, first calculate an angle χ, equalling ψ whenever ψ is a multiple of $\pi/2$ and satisfying

$$\cos\chi = (1-\eta_0^2 \sin^2\psi)^{-\frac{1}{2}}\cos\psi \qquad \sin\chi = (1-\eta_0^2)^{\frac{1}{2}}(1-\eta_0^2 \sin^2\psi)^{-\frac{1}{2}}\sin\psi$$

Then

$$\phi = \beta_3+\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi + B_3\psi + \frac{3}{32}\eta_0^2\,\eta_2^{-4}\sin 2\psi]$$

$$-c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}[A_3 v + \sum_{r=1}^{4} A_{3n}\sin nv]$$

The Zonal Harmonic Perturbations

Within the accuracy of calculation of the perturbations, we may use either the approximate or the accurate formulas listed below. (Use of the approximate formulas may involve more work than use of the accurate ones, because the latter will already be known from the solution of the unperturbed problem.) Compute

	Approximate Formula	Accurate Formula
n	$\mu^{\frac{1}{2}} a^{-3/2}$	
ℓ	$\ell_0 + nt$	$M_s = \ell_0 + 2\pi\nu_1 t$

g	$g_0 + \frac{3}{4}(\frac{r_e}{p})^2 n\ J_2(5\cos^2 I - 1)t$	$g_0 + 2\pi(\nu_2 - \nu_1)t$
E	$M_s + E_0$	$M_s + E_0 + E_1 + E_2$
v	$M_s + v_0$	$M_s + v_0 + v_1 + v_2$

Also, $r = \rho$, within the accuracy of the calculation. Then compute

Third Harmonic Short-Periodic Quantities

$$L_3 = -\frac{nr_e^3}{a}[(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3)\{(\frac{a}{r})^4\sin(v+g) - e(1-e^2)^{-5/2}\sin g\} - \frac{5}{8}\eta_0^3(\frac{a}{r})^4\sin(3v+3g)]$$

$$G_3 = -na^2(1-e^2)^{\frac{1}{2}}(\frac{r_e}{p})^3\{(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3)[e(v-\ell)\cos g + \frac{e^2}{4}\sin(v-g) + (1+\frac{1}{2}e^2)\sin(v+g) + \frac{1}{2}e\sin(2v+g) + \frac{e^2}{12}\sin(3v+g)] - \frac{15}{8}\eta_0^3[\frac{e^2}{4}\sin(v+3g) + \frac{1}{2}e\sin(2v+3g) + \frac{1}{3}(1+\frac{1}{2}e^2)\sin(3v+3g) + \frac{e}{4}\sin(4v+3g) + \frac{e^2}{20}\sin(5v+3g)]\}$$

(Note that $v-\ell = v_0$, within the accuracy of the calculation.)

$$\ell_3 = \frac{3}{8e}(1-e^2)^{\frac{1}{2}}(\frac{r_e}{p})^3\eta_0(5\eta_0^2-4)[(1-e^2)(v-\ell)\sin g + (1+\frac{1}{2}e^2)\cos g + \frac{e}{8}(14-3e^2)\cos(v-g) + \frac{1}{2}e^2\cos(2v-g) + \frac{e^3}{16}\cos(3v-g) - e(1-e^2)\cos(v+g) - \frac{3}{2}\cos(2v+g) - \frac{e}{24}(34-e^2)\cos(3v+g) - \frac{1}{2}e^2\cos(4v+g) - \frac{e^3}{16}\cos(5v+g)] + \frac{5}{8e}(1-e^2)^{\frac{1}{2}}(\frac{r_e}{p})^3\eta_0^3[-\frac{1}{2}e^2\cos 3g - \frac{e^3}{16}\cos(v-3g)$$

$$- \frac{e}{8}(6+5e^2)\cos(v+3g) - (\tfrac{1}{2}+e^2)\cos(2v+3g) + \frac{e}{3}(1-e^2)\cos(3v+3g) + \tfrac{1}{4}(5+e^2)\cos(4v+3g)$$

$$+ \frac{e}{40}(54+e^2)\cos(5v+3g) + \tfrac{1}{2}e^2\cos(6v+3g) + \frac{e^3}{16}\cos(7v+3g)]$$

$$g_3 = -(1-e^2)^{-\frac{1}{2}}\ell_3 - \frac{3}{8}\left(\frac{r_e}{p}\right)^3 \eta_0^{-1}(4-39\eta_0^2+40\eta_0^4)[e(v-\ell)\sin g + \frac{e^2}{4}\cos(v-g)$$

$$-(1+\tfrac{1}{2}e^2)\cos(v+g) - \tfrac{1}{2}e\cos(2v+g) - \frac{e^2}{12}\cos(3v+g)]$$

$$- \frac{5}{8}\left(\frac{r_e}{p}\right)^3 \eta_0(8\eta_0^2-3)[\frac{e^2}{4}\cos(v+3g) + \tfrac{1}{2}e\cos(2v+3g) + \frac{1}{3}(1+\tfrac{1}{2}e^2)\cos(3v+3g)$$

$$+ \frac{e}{4}\cos(4v+3g) + \frac{e^2}{20}\cos(5v+3g)$$

$$h_3 = \left(\frac{r_e}{p}\right)^3 \Big\{ \frac{3}{2}\cot I(1-\frac{15}{4}\eta_0^2)[e(v-\ell)\sin g + \frac{e^2}{4}\cos(v-g) - (1+\tfrac{1}{2}e^2)\cos(v+g)$$

$$-\tfrac{1}{2}e\cos(2v+g) - \frac{e^2}{12}\cos(3v+g)] - \frac{15}{16}\sin 2I[\frac{e^2}{4}\cos(v+3g) + \tfrac{1}{2}e\cos(2v+3g)$$

$$+ \frac{1}{3}(1+\tfrac{1}{2}e^2)\cos(3v+3g) + \frac{e}{4}\cos(4v+3g) + \frac{e^2}{20}\cos(5v+3g)]\Big\}$$

<u>Third Harmonic Long-Periodic Quantities</u>

$$\tilde{G}_3 = \tfrac{1}{2}\, r_e na(1-e^2)^{-\frac{1}{2}}\, e\eta_0 \sin g$$

$$\tilde{\ell}_3 = \frac{r_e}{2ae}(1-e^2)^{\frac{1}{2}}\, \eta_0 \cos g$$

$$\tilde{g}_3 = \frac{r_e}{2p}\left(\frac{e\cos^2 I}{\sin I} - \frac{\sin I}{e}\right)\cos g$$

$$\tilde{h}_3 = -\frac{r_e}{2p}\, e\cot I\cos g$$

Fourth Harmonic Short-Periodic Quantities

$$L_4 = -\frac{1}{8}\left(\frac{r_e}{a}\right)^4(\mu a)^{\frac{1}{2}}\Big\{\left(3-15\eta_0^2+\frac{105}{8}\eta_0^4\right)\left[\left(\frac{a}{r}\right)^5-(1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)\right]$$

$$+5\eta_0^2\left(3-\frac{7}{2}\eta_0^2\right)\left[\left(\frac{a}{r}\right)^5\cos(2v+2g)-\frac{3}{4}e^2(1-e^2)^{-7/2}\cos 2g\right]$$

$$+\frac{35}{8}\eta_0^4\left(\frac{a}{r}\right)^5\cos(4v+4g)\Big\}$$

$$G_4 = -\frac{5}{8}\left(\frac{r_e}{p}\right)^4(\mu p)^{\frac{1}{2}}\eta_0^2\Big\{\left(3-\frac{7}{2}\eta_0^2\right)\left[-\frac{3}{2}e^2(v-\ell)\sin 2g-\frac{e^3}{4}\cos(v-2g)\right.$$

$$+\left(3e+\frac{3e^3}{4}\right)\cos(v+2g)+\left(1+\frac{3}{2}e^2\right)\cos(2v+2g)+\left(e+\frac{e^3}{4}\right)\cos(3v+2g)$$

$$\left.+\frac{3e^2}{8}\cos(4v+2g)+\frac{e^3}{20}\cos(5v+2g)\right]+\frac{7}{8}\eta_0^2\left[\frac{1}{2}e^3\cos(v+4g)+\frac{3e^2}{2}\cos(2v+4g)\right.$$

$$+\left(2e+\frac{1}{2}e^3\right)\cos(3v+4g)+\left(1+\frac{3}{2}e^2\right)\cos(4v+4g)+\left(\frac{6e}{5}+\frac{3e^3}{10}\right)\cos(5v+4g)$$

$$\left.+\frac{1}{2}e^2\cos(6v+4g)+\frac{e^3}{14}\cos(7v+4g)\right]\Big\}$$

$$\ell_{41} = \left(\frac{r_e}{p}\right)^4\frac{(1-e^2)^{\frac{1}{2}}}{2048}\Big\{6(8-40\eta_0^2+35\eta_0^4)\left[48(1-e^2)(v-\ell)\right.$$

$$+\frac{2}{e}(40+12e^2-17e^4)\sin v+4(20-e^2)\sin 2v+e(40-e^2)\sin 3v+10e^2\sin 4v$$

$$\left.+e^3\sin 5v\right]-4\eta_0^2(7\eta_0^2-6)\left[240(1-e^2)(v-\ell)\cos 2g+5e^3\sin(3v-2g)\right.$$

$$+50e^2\sin(2v-2g)+15e(16-3e^2)\sin(v-2g)-20(14+5e^2)\sin 2g$$

$$+\frac{10}{e}(8-24e^2-19e^4)\sin(v+2g)+240(1-e^2)\sin(2v+2g)+\frac{10}{e}(24+16e^2-5e^4)\sin(3v+2g)$$

$$\left.+20(17+2e^2)\sin(4v+2g)+3e(64+e^2)\sin(5v+2g)+50e^2\sin(6v+2g)+5e^3\sin(7v+2g)\right]$$

$$+ \eta_0^4[35e^3\sin(v-4g)-350e^2\sin 4g-105e(8+5e^2)\sin(v+4g)-140(8+11e^2)\sin(2v+4g)$$

$$-\frac{70}{e}(8+20e^2+7e^4)\sin(3v+4g)+840(1-e^2)\sin(4v+4g)+\frac{14}{e}(104+84e^2-13e^4)\sin(5v+4g)$$

$$+140(16+3e^2)\sin(6v+4g)+15e(88+3e^2)\sin(7v+4g)+350e^2\sin(8v+4g)$$

$$+35e^3\sin(9v+4g)]\}$$

$$g_{41} = -(1-e^2)^{-\frac{1}{2}}\ell_{41} - \frac{(r_e/p)^4}{512}\{2(136-500\eta_0^2+385\eta_0^4)[6(2+3e^2)(v-\ell)$$

$$+9e(4+e^2)\sin v+9e^2\sin 2v+e^3\sin 3v]-2(12-82\eta_0^2+77\eta_0^4)[60e^2(v-\ell)\cos 2g$$

$$+10e^3\sin(v-2g)+30e(4+e^2)\sin(v+2g)+20(2+3e^2)\sin(2v+2g)+10e(4+e^2)\sin(3v+2g)$$

$$+15e^2\sin(4v+2g)+2e^3\sin(5v+2g)]+\eta_0^2(11\eta_0^2-4)[35e^3\sin(v+4g)+105e^2\sin(2v+4g)$$

$$+35e(4+e^2)\sin(3v+4g)+35(2+3e^2)\sin(4v+4g)+21e(4+3e^2)\sin(5v+4g)$$

$$+35e^2\sin(6v+4g)+5e^3\sin(7v+4g)]\}$$

$$h_{41} = -\frac{5}{16}\left(\frac{r_e}{p}\right)^4\cos I\{3(7\eta_0^2-4)[(1+\frac{3}{2}e^2)(v-\ell)+(3e+\frac{3}{4}e^3)\sin v+\frac{3}{4}e^2\sin 2v$$

$$+\frac{e^3}{12}\sin 3v]+4(3-7\eta_0^2)[\frac{3}{4}e^2(v-\ell)\cos 2g+\frac{e^3}{8}\sin(v-2g)+(\frac{3e}{2}+\frac{3e^3}{8})\sin(v+2g)$$

$$+(\frac{1}{2}+\frac{3e^2}{4})\sin(2v+2g)+(\frac{1}{2}e+\frac{e^3}{8})\sin(3v+2g)+\frac{3e^2}{16}\sin(4v+2g)+\frac{e^3}{40}\sin(5v+2g)]$$

$$+7\eta_0^2[\frac{e^3}{8}\sin(v+4g)+\frac{3e^2}{8}\sin(2v+4g)+(\frac{1}{2}e+\frac{e^3}{8})\sin(3v+4g)+(\frac{1}{4}+\frac{3e^2}{8})\sin(4v+4g)$$

$$+(\frac{3e}{10}+\frac{3e^3}{40})\sin(5v+4g)+\frac{e^2}{8}\sin(6v+4g)+\frac{e^3}{56}\sin(7v+4g)]\}$$

Fourth Harmonic Long-Periodic Quantities

$$\tilde{G}_4 = -\frac{5r_e^2 n}{16} e^2(1-e^2)^{-3/2}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\cos 2g$$

$$\tilde{\ell}_4 = \frac{5}{16}\left(\frac{r_e}{a}\right)^2(1-e^2)^{-\frac{1}{2}}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\sin 2g$$

$$\tilde{g}_4 = -\frac{5}{32}\left(\frac{r_e}{p}\right)^2[2+e^2-3(2+3e^2)\cos^2 I - 8(2+5e^2)\cos^4 I(1-5\cos^2 I)^{-1}$$

$$-80e^2\cos^6 I(1-5\cos^2 I)^{-2}]\sin 2g$$

$$\tilde{h}_4 = -\frac{5}{16}\left(\frac{r_e}{p}\right)^2 e^2\cos I[3+16\cos^2 I(1-5\cos^2 I)^{-1}+40\cos^4 I(1-5\cos^2 I)^{-2}]\sin 2g$$

Fourth Harmonic Secular Quantities

$$\ell_{42} = -\frac{45}{128}\left(\frac{r_e}{p}\right)^4 n(1-e^2)^{\frac{1}{2}}(3-30\cos^2 I+35\cos^4 I)t$$

$$g_{42} = -\frac{15}{128}\left(\frac{r_e}{p}\right)^4 n[3(4+3e^2)-18(8+7e^2)\cos^2 I+7(28+27e^2)\cos^4 I]t$$

$$h_{42} = -\frac{15}{32}\left(\frac{r_e}{p}\right)^4 n(2+3e^2)\cos I(3-7\cos^2 I)t$$

With $\sigma_4 \equiv J_4 + J_2^2$, next compute the variations in the Delaunay variables

$$\delta L = J_3 L_3 + \sigma_4 L_4 \qquad\qquad \delta H = 0$$

$$\delta G = J_3 G_3 + \frac{J_3}{J_2}\tilde{G}_3 + \sigma_4 G_4 + \frac{\sigma_4}{J_2}\tilde{G}_4$$

$$\delta \ell = J_3 \ell_3 + \frac{J_3}{J_2}\tilde{\ell}_3 + \sigma_4(\ell_{41} + \ell_{42}) + \frac{\sigma_4}{J_2}\tilde{\ell}_4$$

$$\delta g = J_3 g_3 + \frac{J_3}{J_2}\tilde{g}_3 + \sigma_4(g_{41}+g_{42}) + \frac{\sigma_4}{J_2}\tilde{g}_4$$

$$\delta h = J_3 h_3 + \frac{J_3}{J_2}\tilde{h}_3 + \sigma_4(h_{41}+h_{42}) + \frac{\sigma_4}{J_2}\tilde{h}_4$$

The variations of the elements a,e, and η_0 are then

$$\delta a = \frac{2}{an}\,\delta L \qquad \delta e = \frac{pn}{\mu e}\,\delta L - (ae)^{-1}\left(\frac{p}{\mu}\right)^{\frac{1}{2}}\delta G \qquad \delta\eta_0 = \frac{1-\eta_0^2}{\mu\eta_0 p}\,\delta G$$

The variations in the uniformising variables E, v, and ψ are

$$\delta E = (\rho/a)(\delta\ell + \sin E\,\delta e)$$

$$\delta v = (1-e^2)^{\frac{1}{2}}(\rho/a)[\delta E + (1-e^2)^{-1}\sin E\,\delta e]$$

$$\delta\psi = \delta v + \delta g$$

The variations in the spheroidal coordinates ρ, η, and ϕ are then

$$\delta\rho = (\rho/a)\,\delta a - a\cos E\,\delta e + ae\sin E\,\delta E$$

$$\delta\eta = (\eta/\eta_0)\,\delta\eta_0 + \eta_0\cos\psi\,\delta\psi$$

$$\delta\phi = \delta h + (1-\eta_0^2\sin^2\psi)^{-1}\cos I[\delta\psi - \tfrac{1}{2}(\mu p)^{-\frac{1}{2}}\sin 2\psi\,\delta G]$$

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The final rectangular coordinates $X + \delta X$, $Y + \delta Y$, and $Z + \delta Z$ are then given by

$$X + \delta X + i(Y + \delta Y) = [(\rho + \delta\rho)^2 + c^2]^{\frac{1}{2}}[1-(\eta+\delta\eta)^2]^{\frac{1}{2}}\exp i(\phi + \delta\phi)$$

$$\delta Z = \rho\,\delta\eta + \eta\,\delta\rho$$

12. Discussion of Results

Since the ~~necessary~~ chosen accuracy of the quantities appearing in the present perturbation with factor of order J_2^2, ~~theory~~ is only that of an elliptic approximation, the variations in the Delaunay variables may be ~~equated to~~ checked against those found by Brouwer (1959) Garfinkel (1959), or by Kozai (1962). Comparison of the above results with those of Brouwer or of Garfinkel shows that the long-periodic effects of the third and fourth harmonics and the secular effects of the fourth agree with Brouwer's or Garfinkel's, provided that one replaces ~~his~~ their J_4 by $J_4+J_2^2$. Comparison with Kozai shows that the short-periodic effects agree with his.(1) Similarly one can read out of Kozai's paper the long-periodic effects of J_5, J_7, and J_9 and the long-periodic and secular effects of J_6 and J_8; in so doing one ought in principle to replace his J_6 by $J_6-J_2^3$ and his J_8 by $J_8+J_2^4$, but this would be going beyond the accuracy of the present calculation. Since the author's orbital elements differ from Kozai's by terms of order J_2, the agreement with Kozai holds only through terms of order J_2 for long-periodic effects and through terms of order J_2^2 for short-periodic effects.

1. On his page 451, however, in the first line for ΔG, the expression cos2g should be cosg.

To compare accuracies, we construct the following table, noting that the author's reference orbit accounts for about 99.5% of the deviation of the earth's potential from spherical symmetry. Thus my potential perturbation is only about 0.5% of Kozai's ~~perturbation~~.

Effects of 99.5% of Deviation from Sphericity

	Secular Accuracy	Short-Periodic Accuracy	Long-Periodic Accuracy
Kozai	Through J_2^3	Through J_2^2	Terms do not exist
Author	Exact	Through J_2^2	Terms do not exist

Effects of Remaining 0.5% of Deviation from Spherity

	Secular Accuracy	Short-Periodic Accuracy	Long-Periodic Accuracy
Kozai	Through J_2^3	Through J_2^2	Through J_2^2
Author	Through J_2^2	Through J_2^2	Through J_2

Thus the advantages of the author's treatment are the exact solution for the secular effects arising from 99.5% of the aspherical deviation and the much shorter algorithm. The principal advantage of Kozai's treatment, arising in connection with the remaining 0.5% of the aspherical deviation, is his more accurate solution for the long-periodic terms.

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The present solution, like all previous perturbation theories, gives rise to the resonance denominator $1-5\cos^2 I$ in some of the long-periodic terms. These terms are thus not reliable if one considers inclinations I sufficiently close to $63.4°$ or $116.6°$. For such inclinations one could improve the accuracy by boldly dropping the long-periodic terms with coefficient $J_4+J_2^2$ or, better, by superposing on the present treatment Izsak's(1962) solution of the problem of the critical inclination.

The element e occurs in the denominators of δe, ℓ_3, g_3, $\tilde{\ell}_3$, $\tilde{g}_3$ ℓ_{41}, and g_{41} and thus also in the denominators of $\delta\ell$, δg, δE, δv, and $\delta\psi$. No corresponding trouble occurs in the coordinates, however. To test this point, reject all terms except those containing e^{-1}. One then finds $\ell_3 = -g_3$, $\tilde{\ell}_3 = -\tilde{g}_3$, $\ell_{41} = -g_{41}$, $L_3 = G_3$, $L_4 = G_4$, and $\tilde{G}_3 = \tilde{G}_4 = 0$. Then $\delta e = O(e^0)$, $\delta E = \delta v = \delta\ell$, and $\delta\psi = \delta v + \delta g = \delta\ell + \delta g = 0$. Trouble could occur in $\delta\eta$ or in $\delta\phi$ only through $\delta\psi$, so that $\delta\eta$ and $\delta\phi$ do not become infinite for e=0. Similarly, trouble could occur in $\delta\rho$ through the term $-a\cos E\,\delta e$, which, however, does not become infinite, or through the term $ae\sin E\,\delta E$. But $\delta E = \delta\ell = O(e^{-1})$, so that this term also remains finite.

The quantity $\eta \equiv \sin I$ occurs in the denominators of $\delta\eta_0$, g_3, h_3, $\tilde{g}_3$, and $\tilde{h}_3$, and thus also in the denominators of δg, δh, $\delta\psi$, and $\delta\phi$. Again, however, the coordinates remain finite when $\sin I = 0$. To test this point, reject all terms except those containing $\eta_0^{-1} = \csc I$.

Since G_3 and $\widetilde{G}_3$ are of order η_0 and G_4 and $\widetilde{G}_4$ of order η_0^2, it follows that δG is also of order η_0 and $\delta\eta_0 = (1-\eta_0^2)\,\delta G/(\eta_0\mu p)$ remains finite. Thus the term $\sin\psi\,\delta\eta_0$ in $\delta\eta$ remains finite. We also have $g_3 = -h_3 = O(\eta_0^{-1})$, $\widetilde{g}_3 = \frac{1}{2}\, er_e p^{-1}\cos^2 I\csc I\cos g$, $\widetilde{h}_3 = -\frac{1}{2} er_e p^{-1}\cos I\csc I\cos g$, and $\delta\psi = \delta g = O(\eta_0^{-1})$. Thus the term $\eta_0\cos\psi\,\delta\psi$ in $\delta\eta$ also remains finite, so that $\delta\eta$ remains finite. Finally, $\delta\phi = \delta h + \cos I\,\delta g = J_3(g_3+h_3) + J_3J_2^{-1}(\widetilde{h}_3+\widetilde{g}_3\cos I) = \frac{1}{2}\, J_3J_2^{-1}er_e p^{-1}\cos g\csc I(\cos^3 I-\cos I) = -\frac{1}{2}\, J_3J_2^{-1}\, er_e p^{-1}\sin I\cos I\cos g = 0.$

Thus $\delta\phi$ remains finite. There is no trouble with $\delta\rho$.

13. References

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